Baku State University Journal of Mathematics & Computer Sciences ***2025, v. 2 (1), p. 71-81***

journal homepage: http://bsuj.bsu.edu.az/en

ON A NONLINEAR INVERSE BOUNDARY VALUE PROBLEM FOR A LINEARIZED SIXTH-ORDER BOUSSINESQ EQUATION WITH AN ADDITIONAL INTEGRAL CONDITION

Yashar T. Mehraliyev*, Yusif M. Sevdimaliyev, Afaq F. Huseynova

Baku State University,
Received 15 January 2025; accepted 24 February 2025

DOI: **https://doi.org/10.30546/209501.101.2025.3.201.020**

Abstract

We study the classical solution of the nonlinear inverse boundary value problem for for pseudo hyperbolic equation of the fourth order The essence of the problem is that it is required together with the solution to determine the unknown coefficient. The problem is considered in a rectangular area. To solve the considered problem, the transition from the original inverse problem to some auxiliary inverse problem is carried out. The existence and uniqueness of a solution to the auxiliary problem are proved with the help of contracted mappings. Then the transition to the original inverse problem is made, as a result, a conclusion is made about the solvability of the original inverse problem.

Keywords: *inverse boundary value problem, classical solution, uniqueness, existence, Fourier method, Boussinesq equation.*

Mathematics Subject Classification *(2010): 34B09, 34B27, 34L15, 35G31, 35J40, 35R30*

1. Introduction

Let $D_T = \{(x,t):\ 0 \le x \le 1,\quad 0 \le t \le T\}$ and $f(x,t),\ \ \varphi(x)$, $\psi(x)$, $h(t)$ are given functions defined for $x \in [0,1],\ t \in [0,T]$. Consider the following inverse

* * Corresponding author.

E-mail:: ***yashar_aze@mail.ru (Yashar Mehraliyev)***

problem: to find a pair $\{u(x,t),\ a(t)\}$ of the functions $u(x,t),\ a(t)$ satisfying the equation

$$u_{tt}(x,t)-u_{xx}(x,t)+\beta_1 u_{xxxx}(x,t)-\beta_2 u_{xxxxxx}(x,t)=$$
$$=a(t)u(x,t)+f(x,t) \qquad (x,t)\in D_T\ , \tag{1}$$

with initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x) \quad (0\le x\le 1), \tag{2}$$

and boundary conditions

$$u(0,t)=u_x(1,t)=u_{xx}(0,t)=u_{xxx}(1,t)=u_{xxxx}(0,t)=u_{xxxx}(1,t)=0\ (0\le t\le T) \tag{3}$$

and with additional condition

$$\int_0^1 g(x)u(x,t)dx=h(t) \qquad (0\le t\le T), \tag{4}$$

where $\beta_1>0,\quad \beta_2>0$ - are fixed numbers.

Introduce the designation

$$C^{6,2}(D_T)=\{u(x,t):u(x,t)\in C^{2,2}(D_T),u_{xxxxxx}(x,t)\in C(D_T)\}.$$

Definition. A pair $\{u(x,t),a(t)\}$ of the functions $u(x,t)\in C^{6,2}(D_T)$ and $a(t)\in C[0,T]$ satisfying equation (1) in D_T, condition (2) in $[0,1]$ and conditions (3)-(4) in $[0,T]$ we call a classical solution to boundary value (1)-(4).

We prove the following

Теорема1. Let $f(x,t)\in C(\overline{D}_T),\ g(x),\ \varphi(x),\ \psi(x)\in C[0,1],\ h(t)\ne 0\ (0\le t\le T)$ and the matching conditions

$$\int_0^1 g(x)\varphi(x)dx=h(0),\ \int_0^1 g(x)\psi(x)dx=h'(0)\ .$$

are satisfied. Then the problem of finding a classical solution to problem (1)-(4) is equivalent to the problem of determining the functions $u(x,t)\in C^{6,2}(D_T)$ and $a(t)\in C[0,T]$ from (1)-(3) and

$$h''(t)-\int_0^1 g(x)u_{xx}(x,t)dxu+\beta_1\int_0^1 g(x)u_{xxxx}(x,t)dx-\beta_2\int_0^1 g(x)u_{xxxxxx}(x,t)dx=$$

$$= a(t)h(t) + \int_0^1 g(x) f(x,t)dx \ (\ 0 \le t \le T). \qquad (5)$$

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution to problem (1)-(4). Since $h(t) \in C^2[0,T]$, differentiating (4) two times over t we get

$$\int_0^1 g(x)u_t(x,t)dx = h'(t) \ , \ \int_0^1 g(x)u_{tt}(x,t)dx = h''(t) \quad (0 \le t \le T). \qquad (6)$$

We multiply equation (1) by the function $g(x)$ and integrate the resulting equality from 0 to 1 over x, we get:

$$\frac{d^2}{dt^2}\int_0^1 g(x)u(x,t)dx - \int_0^1 g(x)u_{xx}(x,t)dxu + \beta_1\int_0^1 g(x)u_{xxxx}(x,t)dx -$$

$$-\beta_2\int_0^1 g(x)u_{xxxxxx}(x,t)dx = a(t)\int_0^1 g(x)u(x,t)dx + \int_0^1 g(x) f(x,t)dx (\ 0 \le t \le T) \quad (7)$$

From this considering (4) and (6) we arrive at (5).

Now let's suppose that $\{u(x,t), a(t)\}$ is a solution of problem (1)-(3), (5). Then from (5) and (7) we get

$$\frac{d^2}{dt^2}\left(\int_0^1 g(x)u(x,t)dx - h(t)\right) = a(t)\left(\int_0^1 g(x)u(x,t)dx - h(t)\right) \ (0 \le t \le T) \qquad (8)$$

By virtue of (2) and $\int_0^1 g(x)\varphi(x)dx = h(0)$, $\int_0^1 g(x)\psi(x)dx = h'(0)$, we have

$$\int_0^1 g(x)u(x,0)dx - h(0) = \int_0^1 g(x)\varphi(x)dx - h(0) = 0 ,$$

$$\int_0^1 g(x)u_t(0,t)dx - h'(0) = \int_0^1 g(x)\psi(x)dx - h'(0) = 0 . \qquad (9)$$

From (8), taking into account (9), it is clear that condition (4) is also satisfied. The theorem is proved.

2. Solvability of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t), a(t)\}$ to problem (1)-(3), (5) we seek in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x \quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{10}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k=1,2,\ldots).$$

Then applying the formal Fourier scheme, from (1) and (2) we obtain

$$u_k''(t)+(\lambda_k^2+\beta_1\lambda_k^4+\beta_2\lambda_k^6)u_k(t)=F_k(t;u,a) \quad (0\le t\le T; k=1,2,\ldots), \tag{11}$$

$$u_k(0)=\varphi_k,\ u_k'(0)=\psi_k \quad (k=1,2,\ldots), \tag{12}$$

where

$$F_k(t;u,a)=a(t)u_k(t)+f_k(t) \quad ,\ f_k(t)=\int_0^1 f(x,t)\sin\lambda_k x\,dx,$$

$$\varphi_k=2\int_0^1\varphi(x)\sin\lambda_k x dx,\quad \psi_k=2\int_0^1\psi(x)\sin\lambda_k x dx \quad (k=1,2,\ldots).$$

Solving problem (11)-(12) we find

$$u_k(t)=\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k(t-\tau)d\tau \ (k=1,2,\ldots) \tag{13}$$

where

$$\beta_k=\sqrt{\lambda_k^2+\beta_1\lambda_k^4+\beta_2\lambda_k^6} \quad (k=1,2,\ldots).$$

After substitution of the expression $u_k(t)$ $(k=1,2,\ldots)$ into (10) for the determination of $u(x,t)$ we get

$$u(x,t)=$$

$$=\sum_{k=1}^{\infty}\left\{\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k(t-\tau)d\tau\right\}\sin\lambda_k x. \tag{14}$$

Now from (5) taking into account (10) we have

$$a(t)=[h(t)]^{-1}\times$$

$$\times\left\{h''(t)-\int_0^1 g(x)f(x,t)dx+\sum_{k=1}^{\infty}(\lambda_k^2+\beta_1\lambda_k^4+\beta_2\lambda_k^6)u_k(t)\int_0^1 g(x)\sin\lambda_k x dx\right\}. \tag{15}$$

In order to obtain an equation for the second component $a(t)$ of the solution $\{u(x,t),a(t)\}$ of problem (1)-(3), (5) we substitute expression (13) into (15):

$$a(t)=[h(t)]^{-1}\left\{h''(t)-\int_0^1 g(x)f(x,t)dx+\sum_{k=1}^{\infty}(\lambda_k^2+\beta_1\lambda_k^4+\beta_2\lambda_k^6)\left[\varphi_k\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,p)\sin\beta_k(t-\tau)d\tau\right]\int_0^1 g(x)\sin\lambda_k xdx\right\}. \quad (16)$$

Thus, solution of problem (1)-(3),(5) is reduced to the solution of system (14), (16) with respect to the unknown functions $u(x,t)$ and $a(t)$.

To study the problem of the uniqueness of the solution of problem (1)-(3), (5), the following lemma plays an important role.

Lemma. If $\{u(x,t),a(t)\}$ is arbitrary classical solution of problem (1)-(3), (5), then the function

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k xdx \ \ (k=1,2,...)$$

satisfies system (13) in $[0,T]$.

Proof. Let $\{u(x,t),a(t)\}$ be any solution to problem (1)-(3), (5). Then multiplying both sides of equation (1) by the function $2\sin\lambda_k x \ (k=1,2,...)$, integrating the obtained equality over x from 0 to 1 and using the relations

$$2\int_0^1 u_{tt}(x,t)\sin\lambda_k xdx=\frac{d^2}{dt^2}\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=u_k''(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{xx}(x,t)\sin\lambda_k xdx=-\lambda_k^2\left(2\int_0^1 u(x,t)\sin\lambda_k xdx\right)=-\lambda_k^2 u_k(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{xxxx}(x,t)\sin\lambda_k xdx=\lambda_k^4\left(2\int_0^1 u(x,t)\cos\lambda_k xdx\right)=\lambda_k^4 u_k(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{xxxxxx}(x,t)\sin\lambda_k xdx=-\lambda_k^6\left(2\int_0^1 u(x,t)\cos\lambda_k xdx\right)=-\lambda_k^6 u_k(t) \ \ (k=1,2,...)$$

we obtain that equation (11) is satisfied.

Similarly, the fulfilment of (12) is obtained from (2).

Thus $u_k(t)$ $(k=1,2,...)$ is a solution to problem (11), (12). As immediately follows from this the function $u_k(t)$ $(k=1,2,...)$ satisfies to system (13) on $[0,T]$. Lemma is proved.

It is obvious that if $u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx$ $(k=1,2,...)$ is a solution of system (13), then the pair of $\{u(x,t),a(t)\}$ functions $u(x,t)=\sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x$ and $a(t)$ is a solution to system (14), (16).

This lemma implies the validity of the following

Consequence. *Let system (14), (16) have a unique solution. Then problem (1)-(3), (5) cannot have more than one solution, i.e. if problem (1)-(3), (5) has a solution, then it is unique.*

Now, in order to study problem (1)-(3), (5) consider the following spaces.

1. Denote by $B_{2,T}^7$ [15] the set of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty} u_k(t)\sin\lambda_k x \quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right),$$

Defined on D_T, where each of the functions $u_k(t)$ $(k=1,2,\ldots)$ is continuous on $[0,T]$ and

$$J_T(u)\equiv\left(\sum_{k=1}^{\infty}(\lambda_k^7\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this space is defined as

$$\|u(x,t)\|_{B_{2,T}^7}=J(u).$$

2. By E_T^7 we denote the space of the vector functions $\{u(x,t),a(t)\}$ such that $u(x,t)\in B_{2,T}^7$, $a(t)\in C[0,T]$ and equip this space by the norm

$$\|z\|_{E_T^7}=\|u(x,t)\|_{B_{2,T}^7}+\|a(t)\|_{C[0,T]}.$$

Clearly, $B_{2,T}^7$ and E_T^7 are Banach spaces.

Now we consider in E_T^7 the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\},$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,\quad \Phi_2(u,a)=\tilde{a}(t),$$

$\tilde{u}_k(t)$ ($k=1,2,...$) and $\tilde{a}(t)$ are the right hand sides of (13) and (16), correspondingly.

Obviously

$$\varepsilon_1\lambda_k^3\equiv\sqrt{\beta_2}\ \lambda_k^3<\beta_k<\sqrt{1+\beta_1+\beta_2}\ \lambda_k^3\equiv\varepsilon_2\lambda_k^3\ \ (k=1,2,...)\ ..$$

Then we have

$$\left(\sum_{k=1}^{\infty}(\lambda_k^7\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le 2\left(\sum_{k=1}^{\infty}(\lambda_k^7|\varphi_k|)^2\right)^{\frac{1}{2}}+\frac{2}{\varepsilon_1}\left(\sum_{k=1}^{\infty}(\lambda_k^4|\psi_k|)^2\right)^{\frac{1}{2}}+$$

$$+\frac{2\sqrt{T}}{\varepsilon_1}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^4|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\frac{2}{\varepsilon_1}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^7\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \tag{17}$$

$$\|\tilde{a}(t)\|_{C[0,T]}=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-\int_0^1 g(x)f(x,t)dx\right\|_{C[0,T]}+\right.$$

$$+\|g(x)\|_{C[0,1]}(1+\beta_1+\beta_2)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^7|\varphi_k|)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^4|\psi_k|)^2\right)+\right.$$

$$\left.\left.+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^4|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^7\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{18}$$

Assume that the data of problem (1)-(3), (5) satisfy the following conditions:

1. $\varphi(x)\in C^6[0,1],\ \varphi^{(7)}(x)\in L_2(0,1),\ \varphi(0)=\varphi'(1)=\varphi''(0)=\varphi'''(1)=$
$=\varphi^{(4)}(0)=\varphi^{(5)}(1)=\varphi^{(6)}(0)=0$.
2. $\psi(x)\in C^3[0,1],\ \psi^{(4)}(x)\in L_2(0,1),\ \psi(0)=\psi'(1)=\psi''(0)=\psi'''(1)=0$.

3.. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t),\ f_{xxx}(x,t) \in C(D_T),\ f_{xxxx}(x,t) \in L_2(D_T),$

$f(0,t) = f_x(1,t) = f_{xx}(0,t) = f_{xxx}(1,t) = 0\ \ (0 \le t \le T)$.

4. $\beta_1 > 0, \beta_2 > 0, g(x) \in C[0,1],\ h(t) \in C^2[0,T]\ , h(t) \ne 0\ (0 \le t \le T)$.

Then from (17)-(18) we have

$$\|\tilde{u}(x,t)\|_{B^7_{2,T}} \le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^7_{2,T}}, \tag{19}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^7_{2,T}}, \tag{20}$$

where

$$A_1(T) = 2\|\varphi^{(7)}(x)\|_{L_2(0,1)} + \frac{2}{\varepsilon_1}\|\psi^{(4)}(x)\|_{L_2(0,1)} + + \frac{2\sqrt{T}}{\varepsilon_1}\|f_{xxxx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = (1 + \frac{\sqrt{5}}{\varepsilon_1} + T)T\ ,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - \int_0^1 g(x) f(x,t)dx\right\|_{C[0,T]} + \|g(x)\|_{C[0,1]}(1 + \beta_1 + \beta_2) \times\right.$$

$$\left. \times \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\|\varphi^{(7)}(x)\|_{L_2(0,1)} + \|\psi^{(4)}(x)\|_{L_2(0,1)} + \sqrt{T}\|f_{xxxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\|g(x)\|_{C[0,1]}((1 + \beta_1 + \beta_2)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} T.$$

From inequalities (19)-(20) we conclude

$$\|\tilde{u}(x,t)\|_{B^7_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^7_{2,T}}, \tag{21}$$

where

$$A(T) = A_1(T) + A_2(T),\quad B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem:

Theorem 2. Let conditions 1-4 be satisfied and

$$(A(T) + 2)^2 B(T) < 1. \tag{22}$$

The problem (1)-(3),(5) has a unique solution in the ball

$K = K_R(\|z\|_{E_T^5} \le R = A(T)+2)$ of the space E_T^7.

Proof. In the space E_T^7 consider the equation

$$z = \Phi z, \tag{23}$$

where $z = \{u, a\}$, the components $\Phi_i(u,a)$ $(i = 1,2)$ of the operator $\Phi(u,a)$ are defined by the right hand sides of equations (14) and (16) .

Consider the operator $\Phi(u,a)$ in the ball $K = K_R$ from E_T^7. Similarly to (22) we obtain that the estimations

$$\|\Phi z\|_{E_T^7} \le A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^7}, \tag{24}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^7} \le$$
$$\le B(T)R\left(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^7}\right). \tag{25}$$

for the arbitrary $z, z_1, z_2 \in K_R$. Then, from estimates (24), (25), taking into account (22), it follows that the operator Φ acts in the ball and is contractive. Therefore in the ball $K = K_R$ the operator Φ has a single fixed point $\{u, a\}$ which is a unique solution to equation (23) in the ball $K = K_R$, i.e. $\{u, a\}$ is a unique solution to system (14)-(16) in the ball $K = K_R$.

The function $u(x,t)$ as an element of the space $B_{2,T}^7$, has continuous derivatives

$u(x,t),\ u_x(x,t),\ u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t), u_{xxxxx}(x,t), u_{xxxxxx}(x,t)$ in D_T.

As one can easily see from

$$\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le (1+\beta_1+\beta_2)\left(\sum_{k=1}^{\infty}(\lambda_k^7\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$
$$+\left\|\|f_x(x,t) + a(t)u_x(x,t) + b(t)u_{tx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

It implies that $u_{tt}(x,t)$ are continuou in D_T.

It is easy to check that equation (1) and conditions (2), (3) and (5) are satisfied in the usual sense. Therefore, $\{u(x,t), a(t)\}$ is a solution to problem (1)-(3), (5), and, by virtue of the corollary of Lemma 1, it is unique in the ball

$K = K_R$.

The theorem is proved.

Using Theorem 1, we prove the following

Theorem 3. Let all conditions of Theorem 2 be satisfied and

$$\int_0^1 g(x)\varphi(x)dx = h(0), \int_0^1 g(x)\psi(x)dx = h'(0) .$$

The problem (1)-(4) has unique classical solution in the ball $K = K_R(\|z\|_{E_T^5} \le R = A(T)+2)$ from E_T^7 .

References

[1] Tikhonov AN. On the stability of inverse problems. Dokl. USSR Academy of Sciences, **1943**, 39 (5), p.195-198.

[2] Ivanov VK. Linear incorrect problems. DAN SSSR, **1962**, 145(2), 270-272.

[3] Lavrent'ev MM, Romanov VG, Shishatsky ST, Ill-Posed Problems of Mathematical Physics and Analysis, M. Nauka, **1980** (in Russian).

[4] Denisov AM. Introduction to Theory of Inverse Problems, M: MSU, **1994**.

[5] Mehraliyev YT, Huseynova AF. On solvability of an inverse boundary value problem for pseudo hyperbolic equation of the fourth order, Journal of Mathematics Research, **2015**, 7(2), p.101-109.

[6] Esfahani A. and Farah L. Local well-posedness for the sixth-order Boussinesq equation, J. Math. Anal. Appl., 385(**2012**), 230-242.

[7] Megraliev YT, Alizade FH. An inverse problem for a fourth-order Boussinesq equation with non-conjugate boundary and integral overdetermination condition, Herald of Tver State University. Ser. Appl. Math., 2(**2017**), 17-36.

[8] Tekin İ. Reconstruction of a time-dependent potential in a pseudo-hyperbolic equation, U.P.B. Sci. Bull., Series A, 81(**2019**), 115-124.

[5] Tekin İ. Existence and uniqueness of an inverse problem for nonlinear Klein-Gordon equation, Math. Meth. Appl. Sci., 42(**2019**), 3739-3753.

[9]Wang Y. Cauchy problem for the sixth-order damped multidimensional Boussinesq equation, Elec. J. Di . Eqn., 64(**2016**), 1-16.

[10] Wang HW, Esfahani A. Global rough solutions to the sixth-order Boussinesq equation, Nonlinear Anal.-Theor., 102(**2014**), 97-104

[11] Yang He. An inverse problem for the sixth-order linear Boussinesq-type equation. U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 2, **2020**. Pp. 27-36.

[12] Farajov AS, Huntul MJ, Mehraliyev YT. Inverse problem of determining the coefficient in a sixth order Boussinesq equation with additional nonlocal integral condition- Quaestiones Mathematicae, **2024**, v.47(6).pp.1133-1156.

[13] Mehraliyev YT, Farajov AS. On solvability of an inverse boundary value problem for double dispersive sixth order Boussinesq equation. Fundamental and applied problems mathematics and computer science, Materials of the XIV International Conference. Makhachkala, September **2021**. 157-160.

[14] Khudaverdiev KI, Veliyev AA. Study of a one-dimensional mixed problem for a class of third-order pseudohyperbolic equations with a nonlinear operator right-hand side. Baku, **2010**, 168 p.