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ESTIMATES OF THE A. MARCHAUD TYPE FOR PARTIAL AND MIXED MODULI OF SMOOTHNESS OF FRACTIONAL ORDER IN THE CASE OF FUNCTIONS OF TWO VARIABLES

Fuad A. Abdullayev*

Baku State University, Baku, Azerbaijan

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Abstract

In this paper, we study the properties of partial and mixed moduli of smoothness of fractional order in the case of functions with two variables that are 2π –periodic in each variable. Estimates of the type of the Marchaud estimate for the above characteristics of functions with two variables are proved.

Keywords: partial modulus of smoothness, mixed modulus of smoothnes, Cesaro numbers, Marchaud-type estimates.

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1. Introduction

Denote by C_{T^2} the space of continuous functions on

* *Email address: fuadabdullayev@bsu.edu.az*

$T^2 = [-\pi, \pi] \times [-\pi, \pi]$ and 2π – periodic in each of the variables, with the norm

$$\|f\|_C = \max_{(x,y)\in T^2} |f(x,y)|.$$

Let us introduce the following notations (see [2]):

$$\omega_f^{r,\rho}(\delta,\eta) = \sup_{|h_1|\le\delta, |h_2|\le\eta} \left\| \Delta_{h_1,h_2}^{r,\rho} f(x,y) \right\|_C, \ \delta,\eta \in (0,\pi]$$

- mixed modulus of smoothness of the order $r > 0$ with respect to the first argument, the order $\rho > 0$ with respect to the second argument;

$$\omega_f^{r,0}(\delta) = \sup_{|h|\le\delta} \left\| \Delta_h^{r,0} f(x,y) \right\|_C, \ \delta \in (0,\pi]$$

- partial modulus of smoothness of order $r > 0$ with respect to the first argument;

$$\omega_f^{0,\rho}(\eta) = \sup_{|h|\le\eta} \left\| \Delta_h^{0,\rho} f(x,y) \right\|_C, \ \eta \in (0,\pi]$$

- partial modulus of smoothness of order $\rho > 0$ with respect to the second argument, where r, ρ arbitrary positive numbers and

$$\Delta_h^{r,0} f(x,y) = \exp(\pi r i) \sum_{j=0}^{\infty} A_j^{-r-1} f(x + jh, y), \quad (1)$$

$$\Delta_h^{0,\rho} f(x,y) = \exp(\pi \rho i) \sum_{j=0}^{\infty} A_j^{-\rho-1} f(x, y + jh), \quad (2)$$

$$\Delta_{h_1,h_2}^{r,\rho} f(x,y) = \exp(\pi(r+\rho)i) \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} A_j^{-r-1} A_m^{-\rho-1} f(x + jh_1, y + jh_2), \quad (3)$$

and we suppose that $\Delta_{h_1,h_2}^{0,0} f(x,y) = f(x,y)$.

In formulas (1) - (3), the coefficients A_j^{-r-i} and $A_j^{-\rho-1}$ are determined from the relations

$$(1-x)^r = \sum_{j=0}^{\infty} A_j^{-r-1} x^j, \ (1-x)^\rho = \sum_{j=0}^{\infty} A_j^{-\rho-1} x^j. \quad (4)$$

It is obvious that

$$\sum_{j=0}^{\infty} A_j^{-r-1} = 0,\ (1-x)^{\rho} = \sum_{j=0}^{\infty} A_j^{-\rho-1} = 0, \qquad (5)$$

where $r > 0$, $\rho > 0$.

The numbers A_n^{α} are called Cesaro numbers of order α. They have an explicit representation (see [3]):

$$A_n^{\alpha} = \frac{(\alpha+1)\cdots(\alpha+n)}{n!} = \binom{n+\alpha}{n} = O(n^{\alpha}),\ (\alpha \neq -1,-2,\ldots). \qquad (6)$$

If $r > 0$, $\rho > 0$ are integers, then the differences (1) - (3) become ordinary differences of integer order, since $j \geq r+1$, $m \geq \rho+1$, $A_j^{-r-1} = 0$, $A_j^{-\rho-1} = 0$.

From (6) it follows that

$$\sum_{j=0}^{\infty} \left|A_j^{-r-1}\right| < \infty,\quad \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \left|A_j^{-r-1} A_m^{-\rho-1}\right| < \infty. \qquad (7)$$

In particular, It follows from here that the series (1) – (3) converge in the space C_{T^2} :

$$\left\| \sum_{j=n}^{\infty} A_j^{-r-1} f(x+jh, y) \right\|_C \leq \sum_{j=n}^{\infty} \left|A_j^{-r-1}\right| \cdot \|f\|_C = \|f\|_C \cdot \sum_{j=n}^{\infty} \left|A_j^{-r-1}\right| \leq C \cdot n^{-r} \|f\|_C \to 0 \quad ,$$

$n \to \infty$;

$$\left\| \sum_{j=n}^{\infty} \sum_{m=k}^{\infty} A_j^{-r-1} A_m^{-\rho-1} f(x+jh_1, y+mh_2) \right\|_C \leq C \cdot n^{-r} k^{-\rho} \|f\|_C \to 0,\ n \to \infty,\ k \to \infty \cdot$$

In this paper, we study the properties of partial and mixed moduli of smoothness of fractional order in the case of functions with two variables that are 2π–periodic in each variable. Estimates of the type of the Marchot estimate for the above characteristics of functions with two variables are proved

2. Main results

Let's first establish some simple properties of the functions $\omega_f^{r,\rho}(\delta,\eta)$, $\omega_f^{r,0}(\delta)$ and $\omega_f^{0,\rho}(\eta)$.

Property 1. For any $r > 0$, $\rho > 0$

$$\omega_{f+g}^{r,\rho}(\delta,\eta) \le \omega_f^{r,\rho}(\delta,\eta) + \omega_g^{r,\rho}(\delta,\eta). \quad (8)$$

Proof is obvious.

Property 2. For any $r > 0$, $\rho > 0$ the following inequalities hold true:

$$\omega_f^{r,\rho}(\delta,\eta) \le C\|f\|_C; \quad (9)$$

$$\forall 0 \le r_1 \le r,\ 0 \le \rho_1 \le \rho,\ \omega_f^{r,\rho}(\delta,\eta) \le C\omega_f^{r_1,\rho_1}(\delta,\eta), \quad (10)$$

where and in what follows, C denotes various constants in various inequalities that do not depend on f.

Proof. For $\forall 0 \le r_1 \le r$ and $0 \le \rho_1 \le \rho$, we have

$$\Delta_{h_1,h_2}^{r,\rho} f(x,y) = \Delta_{h_1,h_2}^{r-r_1,\rho-\rho_1}\left(\Delta_{h_1,h_2}^{r_1,\rho_1} f(x,y)\right) =$$

$$= \exp(\pi(r+\rho-r_1-\rho_1)i)\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} A_j^{-(r-r_1)-1} \cdot A_m^{-(\rho-\rho_1)-1} \cdot \Delta_{h_1,h_2}^{r_1,\rho_1} f(x+jh_1, y+mh_2).$$

Hence

$$\left\|\Delta_{h_1,h_2}^{r,\rho} f(x,y)\right\|_C \le C\left\|\Delta_{h_1,h_2}^{r_1,\rho_1} f(x,y)\right\|_C.$$

For $r_1 = \rho_1 = 0$, from the last inequality follows the inequality (9), and for $0 < r_1 \le r$, $0 < \rho_1 \le \rho$ follows the inequality (10). In particular, for $r \ge 1,\ \rho \ge 1$ assuming $r_1 = 1,\ \rho_1 = 1$ in (10), we find that $\omega_f^{r,\rho}(\delta,\eta) \to 0$ for $\delta \to 0, \eta \to 0$.

Property 3. For any $r > 0$, $\rho > 0$ holds the following inequality

$$\omega_f^{r,\rho}(2\delta,\eta) \le C\omega_f^{r,\rho}(\delta,\eta). \quad (11)$$

Proof. In [5], a representation for a function of one variable was proved:

$$\Delta_{2h}^r \varphi(x) = \sum_{j=0}^{\infty} B_j^{-r-1}\Delta_h^r \varphi(x+jh),$$

where coefficients B_j^{-r-1} determined from the expansion $(1+x)^r = \sum_{j=0}^{\infty} B_j^{-r-1} x^j$.

From this, it is clear that there is a representation:

$$\Delta_{2h_1,h_2}^{r,\rho} f(x,y) = \sum_{j=0}^{\infty} B_j^{-r-1} \Delta_{h_1,h_2}^{r,\rho} f(x + jh_1, h_2), \tag{12}$$

where coefficients B_j^{-r-1} determined from the expansion

$$(1+x)^r = \sum_{j=0}^{\infty} B_j^{-r-1} x^j . \tag{13}$$

Using representation (12) and taking into account that the coefficients B_n^{α} satisfy estimation (6), we obtain relation (11).

Property 4. For any $r > 0$ and $k \in N$

$$\omega_f^{r,\rho}(k\delta,\eta) \le C\omega_f^{r,\rho}(\delta,\eta). \tag{14}$$

This property is proved by using the representation

$$\Delta_{kh_1,h_2}^{r,\rho} f(x,y) = \sum_{j=0}^{\infty} C_j^{-r-1} \Delta_{h_1,h_2}^{r,\rho} f(x + jh_1, y), \tag{15}$$

where the coefficients C_j^{-r-1} are determined from the expansion

$$\left(1 + x + \ldots + x^{k-1}\right)^r = \sum_{j=0}^{\infty} C_j^{-r-1} x^j . \tag{16}$$

Remark 1. It is clear from the proofs of Properties 3 and 4 that the following inequalities are also true:

$$\forall r > 0,\ \rho > 0,\ \omega_f^{r,\rho}(\delta,2\eta) \le C\omega_f^{r,\rho}(\delta,\eta), \tag{17}$$

$$\forall r > 0,\ \rho > 0,\ \forall k \in N,\ \omega_f^{r,\rho}(\delta,k\eta) \le C\omega_f^{r,\rho}(\delta,\eta). \tag{18}$$

Property 5. Let $0 < \delta < \delta_1$, $0 < \eta < \eta_1$. Then

$$\delta_1^{-r} \eta_1^{-\rho} \omega_f^{r,\rho}(\delta_1,\eta_1) \le C \cdot \delta^{-r} \eta^{-\rho} \omega_f^{r,\rho}(\delta,\eta). \tag{19}$$

Proof. Let $\frac{\delta_1}{\delta} \le k \le \frac{\delta_1}{\delta} + 1$, $\frac{\eta_1}{\eta} \le p \le \frac{\eta_1}{\eta} + 1$, where $k, p \in N$. Then we have

$$\omega_f^{r,\rho}(\delta_1,\eta_1)\le \omega_f^{r,\rho}(k\delta, p\eta)\le C\cdot \omega_f^{r,\rho}(\delta,\eta)\le C\cdot k^r\cdot p^\rho \omega_f^{r,\rho}(\delta,\eta)\le$$

$$\le C\frac{(\delta_1+\delta)^r}{\delta^r}\cdot\frac{(\eta_1+\eta)^p}{\eta^p}\omega_f^{r,\rho}(\delta,\eta)\le C\cdot 2^{r+\rho}\frac{\delta_1{}^r}{\delta^r}\cdot\frac{\eta_1{}^p}{\eta^p}\omega_f^{r,\rho}(\delta,\eta)$$

From this we obtain the required.

We prove the main theorems using properties 1-5 according to the scheme of work ([1]).

Theorem 1 (an analogue of Marchaud's inequality [1]). Let $f\in C_{T^2}$. Then for any $r>0$, $r_1\ge 1$, $\rho>0$, $0<\delta\le\frac{\pi}{r}$ and $\eta\in(0,\pi]$ the inequality is hold:

$$\omega_f^{r_1,\rho}(\delta,\eta)\le C\cdot\delta^{r_1}\left(\int_\delta^{\frac{\pi}{r}}\frac{\omega_f^{r+1,\rho}(t,\eta)}{t^{r_1+1}}dt+\omega_f^{0,\rho}(\eta)\right). \tag{20}$$

Proof. Let us first consider the case of $r=r_1$. Let $h_1\in\left[0,\frac{\pi}{2r}\right]$. Then we have

$$\left|\Delta_{2h_1,h_2}^{r,\rho} f(x,y)-2^r\Delta_{h_1,h_2}^{r,\rho} f(x,y)\right|=\left|\exp(\pi ri)\left(\sum_{j=0}^{\infty}B_j^{-r-1}\Delta_{h_1,h_2}^{r,\rho} f(x+jh,y)\right)-\right.$$

$$\left.-\sum_{j=0}^{\infty}B_j^{-r-1}\Delta_{h_1,h_2}^{r,\rho}\left[f(x+jh_1,y)-f(x,y)\right]\right|=$$

$$=\left|\sum_{j=0}^{\infty}B_j^{-r-1}\Delta_{h_1,h_2}^{r,\rho}\left(\sum_{\nu=0}^{j}\Delta_{h_1,h_2}^{1,0} f(x+\nu h_1,y)\right)\right|=\left|\sum_{j=0}^{\infty}B_j^{-r-1}\sum_{\nu=0}^{j}\Delta_{h_1,h_2}^{r+1,\rho} f(x+\nu h_1,h_2)\right|\le$$

$$\le C\cdot r\cdot 2^{r-1}\omega_f^{r+1,\rho}(h_1,h_2).$$

From the last relation, it follows that

$$\left|\Delta_{h_1,h_2}^{r,\rho} f(x,y)\right|\le C\cdot\left(\frac{r}{2}\omega_f^{r+1,\rho}(h_1,h_2)+\frac{1}{2^r}\left|\Delta_{2h_1,h_2}^{r,\rho} f(x,y)\right|\right).$$

Hence, for any natural p we obtain

$$\left|\Delta_{h_1,h_2}^{r,\rho} f(x,y)\right| \le C\cdot\left(\frac{r}{2}\omega_f^{r+1,\rho}(h_1,h_2)+\frac{r}{2}\frac{1}{2^r}\omega_f^{r+1,\rho}(2h_1,h_2)+\right.$$

$$\left.+\frac{1}{2^{2r}}\left|\Delta_{4h_1,h_2}^{r,\rho} f(x,y)\right|\right)\le\ldots\le$$

$$\le C\left(\frac{r}{2}\sum_{\nu=0}^{p-1}\frac{\omega_f^{r+1,\rho}\left(2^\nu h_1,h_2\right)}{2^{\nu r}}+\frac{1}{2^{pr}}\left|\Delta_{2^p h_1,h_2}^{r,\rho} f(x,y)\right|\right)\le$$

$$\le C\left(\frac{1}{2^{(p-1)r}}\omega_f^{r,\rho}\left(2^p h_1,h_2\right)+\frac{r}{2}\sum_{\nu=0}^{p-1}\frac{\omega_f^{r+1,\rho}\left(2^\nu h_1,h_2\right)}{2^{\nu r}}\right).$$

Further, taking into account that for any integers $j\ge 0$, $h_1\in\left[0,\frac{\pi}{2r}\right]$ and h, such that $\frac{\pi}{2r}\le 2^p h\le\frac{\pi}{r}$, the following inequality holds:

$$h^r\int_{2^j h_1}^{2^{j+1}h_1}\frac{\omega_f^{r+1,\rho}(t,\eta)}{t^{r+1}}dt\ge C\frac{\omega_f^{r+1,\rho}\left(2^j\cdot h_1,\eta\right)}{2\cdot r\cdot 2^{jr}},$$

we have

$$\left|\Delta_{h_1,h_2}^{r,\rho} f(x,y)\right|\le C\cdot\left(\frac{1}{2^{(p-1)r}}\omega_f^{r,\rho}\left(\frac{\pi}{r},h_2\right)+2r\sum_{j=0}^{p-1}h^r\int_{2^j h_1}^{2^{j+1}h_1}\frac{\omega_f^{r+1,\rho}(t,h_2)}{t^{r+1}}dt\right)\le$$

$$\le C\cdot\left(2rh_1^r\int_{h_1}^{\frac{\pi}{r}}\frac{\omega_f^{r+1,\rho}(t,h_2)}{t^{r+1}}dt+h_1^r\omega_f^{0,\rho}(h_2)\right).$$

Thus, for all $\delta\in\left[0,\frac{\pi}{2r}\right]$ we have the following estimate

$$\omega_f^{r,\rho}(\delta,\eta) \le C \cdot \delta^r \left(\int_{\delta}^{\frac{\pi}{r}} \frac{\omega_f^{r+1,\rho}(t,\eta)}{t^{r+1}} dt + \omega_f^{0,\rho}(\eta) \right).$$

The validity of estimation (20) with $\frac{\pi}{2r} \le \delta \le \frac{\pi}{r}$ is easily proved using (14). Thus, (20) is proved for $r = r_1$. Inequality (20) is obvious for $r > r_1$ and for $r < r_1$ it is proved by successive application of the above procedure. The theorem is proved.

Remark 2. Similarly to the proof of Theorem 1, we can prove the following theorems:

Theorem 2. Let $f \in C_{T^2}$. Then for any $\rho > 0$, $\rho_1 \ge 1$, $0 < \eta \le \frac{\pi}{\rho}$ and $r > 0$, $\delta \in (0, \pi]$, the following inequality holds:

$$\omega_f^{r,\rho_1}(\delta,\eta) \le C \cdot \eta^{\rho_1} \left(\int_{\eta}^{\frac{\pi}{\rho}} \frac{\omega_f^{r,\rho+1}(\delta,t)}{t^{\rho_1+1}} dt + \omega_f^{r,0}(\delta) \right). \tag{21}$$

Theorem 3. Let $f \in C_{T^2}$. Then for any $r > 0$, $r_1 \ge 1$, $0 < \delta < \frac{\pi}{r}$, the inequality

$$\omega_f^{r_1,0}(\delta) \le C \cdot \delta^{r_1} \left(\int_{\delta}^{\pi} \frac{\omega_f^{r+1,0}(t)}{t^{r_1+1}} dt + \|f\|_C \right) \tag{22}$$

holds.

Theorem 4. Let $f \in C_{T^2}$. Then for any $\rho > 0$, $\rho_1 \ge 1$, $0 < \eta < \frac{\pi}{\rho}$ and $r > 0$, $\delta \in (0, \pi]$, the inequality

$$\omega_f^{0,\rho_1}(\eta) \le C \cdot \eta^{\rho_1} \left(\int_{\eta}^{\pi} \frac{\omega_f^{0,\rho+1}(t)}{t^{\rho_1+1}} dt + \|f\|_C \right) \tag{23}$$

holds.

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