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POSITIVE AND NEGATIVE SOLUTIONS OF SOME NONLINEAR PROBLEM FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

In this paper, we consider a nonlinear problem for elliptic partial differential equations which dependent on a parameter. The interval of this parameter is determined in which there are positive and negative solutions to the considered nonlinear problem.

Keywords: nonlinear problem, indefinite weight, global bifurcation, continua of silutions

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1. Introduction

Let Ω be a bounded domain in $R^N, N>1$, with a smooth boundary $\partial\Omega$, Let L be the differential operator defined as follows:

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u.$$

We assume that $a_{ij}(x) \in C^1(\overline{\Omega};R),\ i,j=1,2,\ldots,n,\ \ c(x)\in C\left(\overline{\Omega};[0,+\infty)\right)$ and L is uniformly elliptic in $\overline{\Omega}$.

We consider the following nonlinear problem

$$\cdot\begin{cases} Lu(x) = \chi\, a(x)\, g(u(x)),\ x\in\Omega, \\ u=0,\ \ x\in\partial\Omega. \end{cases} \tag{1}$$

Here $a\in C(\overline{\Omega};R)$ and changes sign in Ω, and $g\in C(R;R)$ that satisfy the following condition: there exist nonzero p constants g_0 and g_∞ such that

$$\lim_{|s|\to 0}\frac{g(s)}{s} = g_0 \ \text{ and } \ \lim_{|s|\to +\infty}\frac{g(s)}{s} = g_\infty. \tag{2}$$

Problems of type (1) arise in various fields of physics, namely in the theory of nonlinear diffusion created by nonlinear sources, in the theory of thermal initiation of gases, in quantum field theory and mechanical statistics, and in the theory of gravitational equilibrium of stars (see, for example, [5, 7, 9, 20]).

The existence of positive solutions of nonlinear boundary value problems for elliptic partial differential equations in various formulations has been investigated in many papers (see, for example, [2-7, 9, 11-20]). In these papers, using various methods (analytical methods, a priori estimates, variational methods, degree theory, super-subsolution methods and bifurcation techniques), the existence and multiplicity of positive solutions to the problems under consideration were proved.

It should be noted that the study of the existence of positive solutions to nonlinear boundary value problems for elliptic partial differential equations depending on a parameter has apparently not yet been considered. In this paper, using analytical methods and bifurcation techniques we show the existence of positive and negative solutions of problem (1) depending on a parameter χ.

2. Preliminary

Let $\alpha \in (0, 1)$ be given. We choose $p > 0$ sush that the relations

$$p > n \text{ and } \alpha < 1 - n/p$$

are satisfied. Then It follows from [1, Theorem 6.2, Part III] that $W_2^p(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$.

By E we denote the Banach space $\{u \in C^{1,\alpha}(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$ with the norm $\|\cdot\|_{C^{1,\alpha}}$. We will call u a solution to problem (1) if $u \in W_2^p(\Omega)$ and it satisfies problem (1). Since $W_2^p(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$, any solution of problem (1) belongs to E.

Let P^+ be the set of functions $u \in E$ which satisfy the conditions: $u > 0$ in Ω and $\nu \dfrac{\partial u}{\partial n} < 0$ in $\partial\Omega$, where $\dfrac{\partial u}{\partial n}$ is the outward normal derivative of the function u on $\partial\Omega$. The sets P^+, P^- and $P = P^+ \cup P^-$ are open subsets of E. It is obvious that if $u \in \partial P^\nu$, $\nu \in \{+, -\}$, then either there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, or there exists $x_1 \in \partial\Omega$ such that $\dfrac{\partial u(x_1)}{\partial n} = 0$ (see [4, 14, 15]).

We consider the following spectral problem

$$\begin{cases} Lu = \lambda\, a(x)\, u(x), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{3}$$

By the classical theorem of Krein-Rutman [10], the smallest eigenvalue λ_1 of the linear spectral problem (2) is positive and simple, and has a corresponding eigenfunction u_1 contained in P.

By the max-min principle [8] the eigenvalue λ_1 of problem (3) is given as follows:

$$\lambda_1 = \inf \left\{ \Re[u] : u \in \overset{o}{W}{}_2^1 \right\}, \tag{4}$$

where $\Re[u]$ is a Rayleigh quotient

$$\Re[u] = \frac{\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) u_{x_i}(x) u_{x_j}(x)\, dx + \int_\Omega c(x) u^2(x) dx}{\int_\Omega a(x) u^2(x) dx}. \tag{5}$$

It follows from (2) that

$$g(s) = g_0 s + \tau_0(s)\, s \quad \text{and} \quad g(s) = g_\infty s + \tau_\infty(s)\, s,\ s \in R, \tag{6}$$

where

$$\tau_0(s) \to 0 \quad \text{as} \quad |s| \to 0 \quad \text{and} \quad \tau_\infty(s) \to 0 \quad \text{as} \quad |s| \to +\infty. \tag{7}$$

By (6) problem (1) takes the following equivalent form

$$\begin{cases} Lu(x) = \chi\, g_0\, a(x) u(x) + \chi\, a(x) \tau_0(u(x)) u(x),\ x \in \Omega, \\ u = 0,\ \ x \in \partial\Omega, \end{cases} \tag{8}$$

or

$$\begin{cases} Lu = \chi\, g_0\, a(x) u + \chi\, a(x)(g_\infty - g_0) u + \chi a(x)\, \tau_\infty(u) u,\ x \in \Omega, \\ u(x) = 0,\ \ x \in \partial\Omega, \end{cases} \tag{9}$$

By (7) for any sufficiently small $\varepsilon > 0$ there exists a sufficiently small $\delta_\varepsilon > 0$ and sufficiently large $\Delta_\varepsilon > 0$ such that

$$\left|\tau_0(s)\right| < \varepsilon, \quad \forall\, s \in R,\ s \neq 0, |s| < \delta_\varepsilon, \tag{10}$$

and

$$\left|\tau_\infty(s)\right| < \varepsilon, \quad \forall\, s \in R,\ s \neq 0, |s| > \Delta_\varepsilon, \tag{11}$$

respectively. Then by (10) we get

$$\frac{|\tau_0(s)\, s|}{|s| + |t|} = \frac{|\tau_0(s)\, s|}{|s|} = |\tau_0(s)| < \varepsilon,\ (s,t) \in R \times R^N,\ s \neq 0,\ |s| + |t| < \delta_\varepsilon.$$

which shows that

$$\tau_0(s)\, s = o(|s| + |t|) \quad \text{as} \quad |s| + |t| \to 0. \tag{12}$$

Remark 1. We can extend $\tau_0(s)$ to $s = 0$ by setting $\tau_0(0) = 0$. Then it follows from (7) that $\tau_0(s) \in C(R;R)$.

In view of (6) we have

$$\tau_\infty(s) = g_0 - g_\infty + \tau_0(s),\ s \in R. \tag{13}$$

Hence $\tau_\infty(s) \in C(R;R)$ and it follows from (10) that

$$|\tau_\infty(s)| < |g_0 - g_\infty| + \varepsilon,\ \forall s \in R,\ |s| < \delta_\varepsilon. \tag{14}$$

Moreover, there exists a positive constant ρ_ε such that

$$|\tau_\infty(s)| \le \rho_\varepsilon,\ \forall s \in R,\ \delta_\varepsilon \le |s| \le \Delta_\varepsilon. \tag{15}$$

Let $k_\varepsilon = \max\{|g_0 - g_\infty| + \varepsilon, \rho_\varepsilon\}$. Then (14) and (15) yield

$$|\tau_\infty(s)| \le k_\varepsilon,\ \forall s \in R,\ |s| \le \Delta_\varepsilon. \tag{16}$$

We choose a sufficiently large positive number $\Delta_\varepsilon^* > \Delta_\varepsilon$ so that the following relation holds:

$$\frac{k_\varepsilon \Delta_\varepsilon}{\Delta_\varepsilon^*} < \varepsilon. \tag{17}$$

Let $(s,t) \in R \times R^N$ which satisfies the relation $|s| + |t| > \Delta_\varepsilon^*$. Then by (16) and (17) we get

$$\frac{|\tau_\infty(s)s|}{|s| + |t|} \le \frac{k_\varepsilon \Delta_\varepsilon}{\Delta_{\varepsilon 1}} < \varepsilon \quad \text{for } |s| \le \Delta_\varepsilon, \tag{18}$$

and by (11) we get

$$\frac{|\tau_\infty(s)s|}{|s| + |t|} < \frac{|\tau_\infty(s)s|}{|s|} = |\tau_\infty(s)| < \varepsilon \quad \text{for } |s| > \Delta_\varepsilon. \tag{19}$$

Thus , it follows from (18) and (19) that

$$\frac{|\tau_\infty(s)s|}{|s| + |t|} < \varepsilon \ \text{ if } \ (s,t) \in R \times R^N,\ |s| + |t| > \Delta_\varepsilon^*,$$

which shows that

$$\tau_\infty(s)s = o(|s| + |t|) \ \text{ as } \ |s| + |t| \to +\infty. \tag{20}$$

3. The existence of positive and negative solutions of problem (1)

In this section we will find intervals for χ, in which there are positive and negative solutions, or more precisely, solutions to problem (1) lying in the sets P^+ and P^-, respectively.

Let $\Re = \{(\lambda, 0) : \lambda \in R\}$.

As norm in the Banach space $R\times E$ we take

$$\|(\lambda,u)\|=\{|\lambda|^2+\|u\|^2_{C^{1,\alpha}}\}^{\frac{1}{2}}.$$

Theorem 1. *If* $g_0 g_\infty>0,\ g_0\neq g_\infty,$ *and the condition*

$$\frac{\lambda_1}{g_0}<\chi<\frac{\lambda_1}{g_\infty}\quad \text{or}\quad \frac{\lambda_1}{g_\infty}<\chi<\frac{\lambda_1}{g_0} \tag{21}$$

holds, then for each $\nu\in\{+,-\}$ *there exists a solution* $u_{1,\nu}$ *of problem (1) such that* $u_{1,\nu}\in P^\nu$.

Proof. We consider the following nonlinear eigenvalue problem

$$\begin{cases} Lu(x)=\lambda\chi\, g_0 a(x)u(x)+\chi a(x)\tau_0(u(x))u(x),\ x\in\Omega,\\ u=0,\ \ x\in\partial\Omega. \end{cases} \tag{22}$$

Let $\chi>0$ be fixed and $g_0>0$. Then, in view of relation (12), by [14, Theorem 2.12 and Corollary 2.13] for each $\nu\in\{+,-\}$ there exists a continuum C_0^ν of nontrivial solutions of problem (22) that meets $(\tilde{\lambda}_1,0)$, lies in $R\times P^\nu$ and is unbounded in $R\times E$, where $\tilde{\lambda}_1$ is a positive smallest eigenvalue of the linear problem

$$\begin{cases} Lu(x)=\lambda\chi\, g_0 a(x)u(x),\ x\in\Omega,\\ u=0,\ \ x\in\partial\Omega. \end{cases} \tag{23}$$

Note that in this case either C_0^ν meets (λ,∞) for some $\lambda\in R,$ or the projection $P_{\Re}(C_0^\nu)$ of C_0^ν on $\Re$ is unbounded.

In view of (3), by (23) we get

$$\tilde{\lambda}_1=\frac{\lambda_1}{\chi g_0}. \tag{24}$$

Using (9) we can rewrite (22) as follows

$$\begin{cases} Lu(x)=\lambda\chi\, g_0 a(x)u(x)+\chi a(x)(g_\infty-g_0)u(x)+\chi\, a(x)\,\tau_\infty(u(x)),\ x\in\Omega,\\ u(x)=0,\ \ x\in\partial\Omega. \end{cases} \tag{25}$$

By (20) it follows from [15, Theorem 2.28 and Corollary 2.37] that for each $\nu\in\{+,-\}$ there is a connected component C_∞^ν of nontrivial solutions of problem (25) (or (22)) which meets $(\hat{\lambda}_1,\infty)$, is contained in $R\times P^\nu$ and either meets $\Re$, or the projection $P_{\Re}(C_{\infty}^{\nu})$ of this set on $\Re$ is unbounded, where $\hat{\lambda}_1$ is a smallest eigenvalue of the linear problem

$$\begin{cases} Lu(x) = \lambda\chi\, g_0 a(x) u(x) + \chi\, a(x)(g_\infty - g_0) u(x),\ x \in \Omega, \\ u(x) = 0,\ \ x \in \partial\Omega. \end{cases} \tag{26}$$

Note that (26) can be rewritten in the form

$$\begin{cases} Lu(x) = \left(\lambda + \dfrac{g_\infty}{g_0} - 1 \right) \chi\, g_0 a(x) u(x),\ x \in \Omega, \\ u(x) = 0,\ \ x \in \partial\Omega. \end{cases} \tag{27}$$

It is clear from (3) and (27) that

$$\left(\hat{\lambda}_1 + \frac{g_\infty}{g_0} - 1 \right) \chi g_0 = \lambda_1,$$

and consequently,

$$\hat{\lambda}_1 = \frac{\lambda_1}{\chi g_0} - \frac{g_\infty}{g_0} + 1 \tag{28}$$

Now we prove that the set $P_{\Re}(C_0^{\nu})$ is bounded. Indeed, otherwise there exists a sequence $\{(\lambda_{k,\nu}, u_{k,\nu})\}_{k=1}^{\infty} \subset R \times P^{\nu}$ of nontrivial solutions of (22) such that

$$\lambda_{k,\nu} \to +\infty \ \text{ as } \ k \to \infty. \tag{29}$$

Let

$$\varphi_{k,\nu}(x) = -\tau_0(u_{k,\nu}(x)), x \in \overline{\Omega}. \tag{30}$$

Then $(\lambda_{k,\nu}, u_{k,\nu})$ for each $k \in \mathrm{N}$ is a solution of the following linear problem

$$\begin{cases} Lu(x) + a(x)\varphi_{k,\nu}(x) u(x) = \lambda\chi\, g_0 a(x) u(x),\ x \in \Omega, \\ u = 0,\ \ x \in \partial\Omega. \end{cases} \tag{31}$$

Let ε_0 be fixed sufficiently small positive number. Then, by (10), (11) and (12), we obtain

$$\left|\tau_0(s)\right| < \varepsilon_0 \ \text{ if } |s| < \delta_{\varepsilon_0}, \ \text{ and } \ \left|\tau_0(s)\right| < |g_0 - g_\infty| + \varepsilon_0 \ \text{ if } \ |s| > \Delta_{\varepsilon_0}. \tag{32}$$

By Remark 1 there exists a positive constant κ_0 such that

$$\left|\tau_0(s)\right| < \kappa_0 \ \text{ if } \ \delta_{\varepsilon_0} \le |s| \le \Delta_{\varepsilon_0}. \tag{33}$$

Then by (32) and (33) we get

$$\left|\tau_0(s)\right| < \tilde{\kappa}_0 \quad \text{for} \quad s \in R,$$

where $\tilde{\kappa}_0 = \max\{\kappa_0, |g_0 - g_\infty| + \varepsilon_0\}$, and consequently, by (30) we obtain

$$|\varphi_{k,\nu}(x)| \le \tilde{\kappa}_0, \ \ x \in \overline{\Omega}. \tag{34}$$

By using the max-min principle (see (4) and (5)) we have

$$\lambda_{k,\nu} = \inf \left\{ \frac{1}{\chi\, g_0} \Re[u] + \frac{1}{g_0} \frac{\int\limits_\Omega \varphi_{k,\nu}(x)\, a(x)\, u^2(x) dx}{\int\limits_\Omega a(x) u^2 dx} : u \in \overset{o}{W}{}_2^1 \right\}. \tag{35}$$

In view of (34) we get

$$\left| \frac{\int\limits_\Omega \varphi_{k,\nu}(x)\, a(x)\, u^2(x) dx}{\int\limits_\Omega a(x) u^2 dx} \right| \le \tilde{\kappa}_0. \tag{36}$$

Then by (36) it follows from (4), (5) and (35) that

$$\frac{\lambda_1}{\chi g_0} - \frac{\tilde{\kappa}_0}{g_0} \le \lambda_{k,\nu} \le \frac{\lambda_1}{\chi g_0} + \frac{\tilde{\kappa}_0}{g_0},$$

which contradicts relation (29). Thus the continuum $C_0^\nu \subset R \times P^\nu$ meets (λ, ∞) for some $\lambda \in R$. In a similar way we can show that $P_\Re(C_\infty^\nu)$ is also bounded. Then, by the above arguments, the continuum $C_\infty^\nu \subset R \times P^\nu$ meets $(\lambda, 0)$ for some $\lambda \in R$. Hence in view of [17, Theorem 3.3] and [15, Corollary 2.39] C_0^ν meets $(\hat{\lambda}_1, \infty)$ and C_∞^ν meets $(\tilde{\lambda}_1, 0)$, which implies that the sets C_0^ν and C_∞^ν coincide. Therefore, C_0^ν can cross the hyper-plane $\{1\} \times E$ in $R \times E$ only in the case when

$$\tilde{\lambda}_1 < 1 < \hat{\lambda}_1 \quad \text{or} \quad \hat{\lambda}_1 < 1 < \tilde{\lambda}_1. \tag{37}$$

Thus, if condition (37) is satisfied, then for each $\nu \in \{+, -\}$ problem (22) has a solution of the form $(1, u_{1,\nu}) \in R \times P^\nu$, which implies that problem (1) has a solution $u_{1,\nu} \in P^\nu$.

Let $g_0 > g_\infty > 0$ and the first condition of (21) holds, i.e.

$$\frac{\lambda_1}{g_0} < \chi < \frac{\lambda_1}{g_\infty}.$$

Then $\frac{\lambda_1}{\chi g_0} < 1$, and consequently, by (24) we get

$$\tilde{\lambda}_1 < 1. \tag{38}$$

Moreover, we have the following relation: $\frac{\lambda_1}{\chi g_\infty} - 1 > 0$. Multiplying this inequality by $\frac{g_\infty}{g_0}$ we obtain

$$\frac{\lambda_1}{\chi g_0} - \frac{g_\infty}{g_0} > 0.$$

Then, by (28), it follows from last relation that

$$\hat{\lambda}_1 = \frac{\lambda_1}{\chi g_0} - \frac{g_\infty}{g_0} + 1 > 1. \tag{39}$$

Relations (38) and (39) are equivalent to the first relation of (37).

The remaining cases are considered similarly. The proof of this theorem is complete.

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