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LINEAR INVERSE PROBLEMS FOR A ONE-DIMENSIONAL SECOND-ORDER PARABOLIC EQUATION TO FIND THE RIGHT-HAND SIDE

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Abstract

This paper investigates the solvability of an inverse boundary value problem with an unknown time-dependent right-hand side for a second-order parabolic equation. First, an original problem is reduced to the equivalent problem, the theorem of existence and uniqueness of solution is proved for the latter. Then, using these facts the author proves existence and uniqueness of classical solution of the original problem.

Keywords: *inverse boundary problem, parabolic equation, Fourier method, classical solution.*

Mathematics Subject Classification *(2020):* 35R30, 35R11; 35A01, 35A02, 35A09

1. Introduction

It is known that the practical requirements often lead to the problem of determining the coefficients or the right hand side of the partial differential equations for some known data about its solutions. Such problems are called inverse boundary value problems in mathematical physics. If the unknowns in an

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inverse problem are the solution and the right-hand side, then such an inverse problem is called linear; if the unknowns are the solution and at least one of the coefficients, then the inverse problem is nonlinear.

Inverse problems are a beneficially developing section of modern mathematics. Recently, inverse problems have been widely used in various fields of science. Inverse problems for various types of partial differential equations have been studied in many works. We note the works of A.N. Tikhonov [1], M.M. Lavrentyev [2,3], A.M. Denisov [4], M.I. Ivanchov [5], S.I. Kabanikhin [6].

The solvability of inverse problems in various formulations with various overdetermination conditions for parabolic equations was the subject of study in [7]–[14].

Unlike known works, in this paper, a linear inverse boundary value problem for a second-order parabolic equation

2. Problem statement and its reduction to an equivalent problem

Consider the equation

$$c(t)u_t(x,t)-u_{xx}(x,t)=F(x,t) \qquad (1)$$

in the domain $D_T=\{(x,t):0<x<1, 0<t\le T\}$ and set up a boundary value problem for it with the initial condition

$$u(x,0)=\varphi(x), \quad (0\le x\le 1), \qquad (2)$$

Neyman boundary condition

$$u_x(0,\ t)=0 \quad (0\le t\le T), \qquad (3)$$

Dirichlet boundary condition

$$u(1,\ t)=0 \quad (0\le t\le T), \qquad (4)$$

where $0<c(t)\in C[0,T]$ $\varphi(x)\in C[0,1]$, $F(x,t)\in C(\bar{D}_T)$ - are the given functions, and $u(x,t)$ is the desired function.

Let's accept the following

Definition 1. *The function $u(x,t)\in C^{2,1}(\bar{D}_T)$ is said to be a classical solution of the problem (1)-(4), if this function satisfy Equation (1) in D_T, the condition (2)* *on* $[0,1]$ *, and the statements (3)-(4) on the interval* $[0,T]$ *.*

Theorem 1. *Assume that* $0 < c(t) \in C[0,T]$ $(0 \le t \le T)$ *. Then if problem (1) - (4) has a solution, then it is unique .*

Proof. Assume that there are two solutions to the considered problem as $u_1(x,t)$ and $u_2(x,t)$. Let us denote the difference of these solutions by $\vartheta(x,t) = u_1(x,t) - u_2(x,t)$. It is clear that the function $\vartheta(x,t)$ satisfies the following homogeneous equation

$$c(t)v_t(x,t) - v_{xx}(x,t) = 0 \quad (0 \le x \le 1, 0 \le t \le T), \tag{5}$$

and the conditions

$$v(x,0) = 0, \quad (0 \le x \le 1), \tag{6}$$

$$v_x(0,t) = 0 \quad (0 \le t \le T), \tag{7}$$

$$v(1,\, t) = 0 \ (0 \le t \le T). \tag{8}$$

Let us prove that the function $v(x,t)$ is identically equal to zero. Obviously $v(x,t) \equiv 0$ is a solution to this problem. Let us prove that there are no other solutions.

Multiplying both sides of equation (5) by the function $2v(x,t)$ and integrate the resulting equality over x from 0 to 1:

$$2\int_0^1 c(t)v_t(x,t)v(x,t)dx - 2\int_0^1 v_{xx}(x,t)v(x,t)dx = 0 \tag{9}$$

Using the boundary conditions (7), (8), and (5) we have:

$$2c(t)\int_0^1 v_t(x,t)v(x,t)dx = c(t)\frac{d}{dt}\int_0^1 v^2(x,t)dx \, (0 \le t \le T);$$

$$2\int_0^1 v_{xx}(x,t))v(x,t)dx = 2v_x(1,t)v(1,t) - 2v_x(0,t)v(0,t) - \int_0^1 v_x^2(x,t)dx =$$

$$= -\int_0^1 v_x^2(x,t)dx;$$

Then, from (9), we get

$$c(t)\frac{d}{dt}\int_0^1 v^2(x,t)dx = -2\int_0^1 v_x^2(x,t)dx \ \ (0 \le t \le T) \qquad (10)$$

We introduce the following notation

$$z(t) = \int_0^1 v^2(x,t)dx \ge 0 \ .$$

It is obvious from (6) that $z(0) = 0,$ and in turn from (10) it is easy to see that $z'(t) \le 0\,(0 \le t \le T)$. From here we get:

$$c(t)\int_0^1 v^2(x,t)dx = 0 \ (0 \le t \le T)\,.$$

Then from the last relation it is easy to see that $v(x,t) = 0 \quad (x,t) \in \bar{D}_T$.

Thus, it is shown that if there exist two solutions and of problem (1)-(4), then.

Thus, it is shown that if there are two solutions $u_1(x,t)$ and $u_2(x,t)$ problem (1)-(4), then $u_1(x,t) \equiv u_2(x,t)$. It follows that if a solution to problem (1)-(4) exists in $u(x,t) \in C^{2,1}(\bar{D}_T)$, then it is unique. The theorem is proved.

Based on the direct problem (1)-(4) we consider the following inverse problem. Let

$$F(x,t) = a(t)g(x,t) + f(x,t) \qquad (11)$$

where the functions $g(x,t)$ and f $f(x,t)$ are given functions and the function $a(t)$ is unknown.

It is required to determine $a(t)$, if the following additional information about the solution of problem (1)-(4) is given:

$$u(0,t) = h(t) \ \ (0 \le t \le T) \qquad (12)$$

the functions $h(t)$ are unknown function.

Definition 2. *A pair* $\{u(x,t), a(t)\}-$*is called a classical solution to problem (1)-(4), (12) if the functions* $u(x,t) \in C^{2,1}(\bar{D}_T)$ *and* $a(t) \in C[0,T]$ *satisfy Equation (1) in* D_T *, the condition (2) in* $[0,1]$ *, the conditions (3), (4) and (12) on the interval* $[0,T]$.

The following theorem is true.

Theorem 2. *Assume that* $f(x,t), g \in (x, \bar{t}_T)$ $\varphi(x) \in C$ [($h(t) \in C^1[0,T]$, $0 < c(t) \in C[0,T]$, $g(0,t) \neq 0$ $(0 \le t \le T)$, *and the compatibility conditions*

$$\varphi(0) = h(0)$$

holds. Then the problem of finding a classical solution of (1)-(4), (12) is equivalent to the problem of determining the functions $u(x,t) \in C^{2,1}(\bar{D}_T)$ *and* $a(t) \in C[0,T]$, *satisfying the conditions (1)-(4) and the relation*

$$c(t)h'(t) - u_{xx}(0,t) = a(t)g(0,t) + f(0,t) \ (0 \le t \le T). \qquad (13)$$

Proof. Let $\{u(x,t), a(t)\}$ – be a classical solution of (1)-(4), (12).

Substituting $x = 0$ into (1) we find:

$$c(t)\frac{d}{dt}u(0,t) - u_{xx}(0,t) = a(t)g(0,t) + f(0,t) \ (\ 0 \le t \le T). \qquad (14)$$

Further, assuming $h(t) \in C^1[0,T]$ and differentiating (12) twice, we have

$$u_t(0,t) = h'(t) \ \ (0 \le t \le T) \qquad (15)$$

From (14), taking into account (15), it follows the fulfillment of (13).

Now, suppose that $\{\{u(x,t), a(t)\}$ is a solution to problem (1)-(4), (13). Then from (13) and (14) we find

$$c(t)\frac{d}{dt}\Big(u(0,t) - h(t)\Big) = 0 \quad (0 \le t \le T). \qquad (16)$$

By virtue of (2) and the conditions of agreement $\varphi(0) = h(0)$, we have:

$$u(x_0, 0) - h(0) = \varphi(x_0) - h(0) = 0 \qquad (17)$$

From (16) and (17) we conclude that condition (12) is satisfied. The theorem has been proven.

3. On the solvability of the inverse boundary value problem

We seek the first component $u(x,t)$ of solution $\{u(x,t), a(t)\}$ of the problem (1)- (4), (13) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \cos \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right), \qquad (18)$$

where

$$u_k(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx \quad (k=1,2,...).$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$c(t)u_k'(t)+\lambda_k^2 u_k(t)=F_k(t;a) \quad (k=1,2,...;0\le t\le T), \tag{19}$$

$$u_k(0)=\varphi_k (k=1,2,...), \tag{20}$$

where

$$F_k(t;a)=a(t)g_k(t)+f_k(t),$$

$$f_k(t)=2\int_0^1 f(x,t)\cos\lambda_k xdx, g_k(t)=2\int_0^1 g(x,t)\cos\lambda_k xdx,$$

$$\varphi_k=2\int_0^1 \varphi(x)\cos\lambda_k xdx \quad (k=1,2,...).$$

Solving problem (19), (20), we find

$$u_k(t)=\varphi_k e^{-\int_0^t \frac{\lambda_k^2 ds}{c(s)}}+\int_0^t \frac{F_k(\tau;a)}{c(\tau)}e^{-\int_\tau^t \frac{\lambda_k^2 ds}{c(s)}}d\tau \, (k=1,2,...). \tag{21}$$

Substituting the expressions $u_k(t),\ k=1,2,...$) described by (21) into (18), to determine the first component of the solution (1)-(4), (13). we obtain

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\varphi_k e^{-\int_0^t \frac{\lambda_k^2 ds}{c(s)}}+\int_0^t \frac{F_k(\tau;a)}{c(\tau)}e^{-\int_\tau^t \frac{\lambda_k^2 ds}{c(s)}}d\tau\right\}\cos\lambda_k x. \tag{22}$$

Now, from (13), taking into account (18), we have:

$$a(t)=\left[g(0,t)\right]^{-1}\left\{h'(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\right\}. \tag{23}$$

Substituting expression (26) into (29) we obtain:

$$a(t)=\left[g(0,t)\right]^{-1}\times$$

$$\times\left\{h'(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_k e^{-\int_0^t \frac{\lambda_k^2 ds}{c(s)}}+\int_0^t \frac{F_k(\tau;a)}{c(\tau)}e^{-\int_\tau^t \frac{\lambda_k^2 ds}{c(s)}}d\tau\right]\right\}. \tag{24}$$

Thus, the solution to problem (1)-(4), (13) is reduced to the solution of system (22), (24) with respect to unknown functions $u(x,t)$ and $a(t)$.

It is easy to see that $a(t)$ is a solution to equation (30), then a pair $\{u(x,t),a(t)\}$ of functions $u(x,t)$ and $a(t)$ will be a solution to problem (1)-(4), (13). Therefore, the problem posed is reduced to determining $a(t)$ from equation (24).

The following theorem is true:

Theorem 3. Let us assume that the data of problems (1)-(4), (13) satisfy the following conditions:

1. $\varphi(x)\in C^2[0,1], \varphi'''(x)\in L_2(0,1), \varphi'(0)=0$, $\varphi(1)=0$, $,\varphi''(1)=0$.
2. $f(x,t), f_x(x,t), f_{xx}(x,t)\in C(D_T),, f_{xxx}(x,t)\in L_2(D_T),$
   $f_x(0,t)=0, \quad f(1,t)=0$, $f_{xx}(1,t)=0 \qquad (0\le t\le T)$.
3. $g(x,t), g_x(x,t), g_{xx}(x,t)\in C(D_T), g_{xxx}(x,t)\in L_2(D_T), g_x(0,t)=0,$
   $g(1,t)=0$, $g_{xx}(1,t)=0 \qquad (0\le t\le T)$.
4. $0<c(t)\in C[0,T], h(t)\in C^1[0,T]$, $g(0,t)\ne 0 (0\le t\le T)$.

Then problem (1)–(4), (13) has a unique solution.

Proof. Equation (24) can be written as

$$a(t)=q(t)+\int_0^t G(t,\tau)a(\tau)d\tau\ , \tag{25}$$

where

$$q(t)=\left[g(0,t)\right]^{-1}\left\{h'(t)-f(0,t)+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_k e^{-\int_0^t\frac{\lambda_k^2 ds}{c(s)}}+\int_0^t\frac{f_k(\tau)}{c(\tau)}e^{-\int_\tau^t\frac{\lambda_k^2 ds}{c(s)}}d\tau\right]\right\},$$

$$G(t,\tau)=\left[g(0,t)\right]^{-1}\sum_{k=1}^{\infty}\frac{\lambda_k^2 e^{-\int_\tau^t\frac{\lambda_k^2 ds}{c(s)}}}{c(\tau)}g_k(\tau)\ .$$

Taking into account the conditions of Theorem 3 it can be shown that

$$\|q(t)\|_{C[0,T]} = \left\| \left[g(0,t) \right]^{-1} \right\|_{C[0,T]} \left\{ \|h'(t) - f(0,t)\|_{C[0,T]} + \right.$$

$$\left. +M \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{T} \left\| [c(t)]^{-1} \right\|_{C[0,T]} \| f_{xxx}(x,t) \|_{L_2(D_T)} \right] \right\}, \quad (26)$$

$$|G(t,\tau)| \le \left\| \left[g(0,t) \right]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| \| g_{xxx}(x,t) \|_{L_2(0,1)} \right\|_{C[0,T]} . \quad (27)$$

From (26) and (27) it follows that the linear integral equation of Volterra type (25 has a unique solution from. $C[0,T]$.

Let us show that if $a(t) \in C[0,T]$, then $u(x,t) \in C^{2,1}(\bar{D}_T)$. Indeed, it is clear from (21) that $u_k(t) \in C[0,T] (k = 1, 2, ..)$ and

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_k(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} \le \sqrt{3} \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_k| \right)^2 \right)^{\frac{1}{2}} + \right.$$

$$+ \left\| [c(t)]^{-1} \right\|_{C[0,T]} \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 | f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} +$$

$$\left. + \sqrt{T} \|a(t)\|_{C[0,T]} \left\| [c(t)]^{-1} \right\|_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |g_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right]$$

or

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \| u_k(t) \|_{C[0,T]})^2 \right)^{\frac{1}{2}} \le \sqrt{3} \left[\|\varphi'''(x)\|_{L_2(0,1)} + \right.$$

$$+ \sqrt{T} \left\| [c(t)]^{-1} \right\|_{C[0,T]} \| f_{xxx}(x,t) \|_{L_2(D_T)} +$$

$$\left. + \sqrt{T} \left\| [c(t)]^{-1} \right\|_{C[0,T]} \|a(t)\|_{C[0,T]} \| g_{xxx}(x,t) \|_{L_2(D_T)} \right]$$

From the last relation it is clear that the function $u(x,t)$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in $\bar{D}_T$.

From equation (25), in view of (21), it is easy to see that

$$\left(\sum_{k=1}^{\infty}(\lambda_k \|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \right.$$

$$\left. + \left\| \| f_x(x,t) + a(t) g_x(x,t) \|_{L_2(0,1)} \right\|_{C[0,T]} \right\}.$$

It follows that $u_t(x,t)$ is continuous in $\bar{D}_T$.

It is easy to verify that equation (1) and conditions (2)–(4), (13) are satisfied in the usual sense. Thus, the solution to problem (1)–(4), (13) is a pair of $\{u(x,t), a(t)\}$ functions. By Theorem 1, it is unique. The theorem is proved.

Using Theorem 2, the last theorem immediately implies the unique solvability of the original problem (1)–(4), (12).

Theorem 4. Let all the conditions of Theorem 3 be satisfied and the consistency conditions be satisfied:

$$\varphi(0) = h(0).$$

Then problem (1)–(4), (12) has a unique classical solution.

CONCLUSION

The unique solvability of a linear inverse boundary value problem for a second-order parabolic equation is investigated. The problem under consideration is, in a certain sense, reduced to an auxiliary problem, and the existence and uniqueness of a solution to the auxiliary problem are proven. Based on the equivalence of these problems, an existence and uniqueness theorem for a solution to the original problem is proved.

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