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ON THE NONLINEAR STURM-LIOUVILLE PROBLEM WITH INDEFINITE WEIGHT AND SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract

In this paper, we consider the nonlinear Sturm-Liouville problem with indefinite weight function and spectral parameter in the boundary condition. We show the existence of four families of continua of solutions corresponding to the usual nodal properties and branching from intervals of the line of trivial solutions and from intervals of $R \times \{\infty\}$.

Keywords: nonlinear Sturm-Liouville problem, indefinite weight, spectral parameter, bifurcation from zero, bifurcation from infinity.

Mathematics Subject Classification (2020): 34B09, 34B15, 34C10, 34C23, 34K11, 34L20, 47J10, 47J15.

1. Introduction

We consider the following nonlinear eigenvalue problem

$$\ell(y) \equiv -(p(x)y')' + q(x)y = \lambda r(x)y + f(x, y, y, \lambda),\ x \in (0, 1), \tag{1}$$

$$b_0 y(0) = d_0 p(0) y'(0), \tag{2}$$

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$$(a_1\lambda + b_1)y(1) = d_1 p(1)y'(1), \tag{3}$$

where $\lambda \in R$ is a parameter, p is a positive and continuously differentiable function on $[0, 1]$, q is a nonnegative continuous function on $[0, 1]$, r is a sign-chaning continuous function on $[0, 1]$, b_0, d_0, a_1, b_1, d_1 are real constants such that $|b_0| + |d_0| > 0$, $b_0 d_0 \geq 0$, and if $b_0 = 0$, then q is not identically zero, and $a_1 d_1 > 0$, $b_1 d_1 \leq 0$. The nonlinear term f is a continuous function on $[0, 1] \times R^3$ satisfying the following conditions:

$$u\, f(x,u,s,\lambda) \leq 0,\ x \in [0, 1],\ (u,s) \in R^2, \lambda \in R; \tag{4}$$

there exist a positive number M, a positive sufficiently small number c_0 and a positive sufficiently large number c_1 such that

$$\left|\frac{f(x,u,s,\lambda}{u}\right| \leq M, x \in [0, 1],\ (u,s) \in R^2, u \neq 0, \\ |u| + |s| \leq c_0, |u| + |s| \geq c_1, \lambda \in R. \tag{5}$$

Bifurcation of solutions to nonlinear Sturm-Liouville problems with indefinite weight arise in the study of various problems in mechanics, physics, biology, ecology and other areas of natural science (see [12, 13, 18, 19] and its bibliography). Note that problem (1)-(3) with $a_1 = 0$ arises when modeling the selection-migration process in population genetics (see [12, 13]).

For nonlinear Sturm-Liouville problems of second and fourth orders with definite weight, the bifurcation of solutions from zero and infinity has been studied in detail since the 70s of the last century in the works of J.F. Toland [25], C.A. Stuart [24], P.H. Rabinowitz [21, 22], H. Berestycki [9], B.P. Rynne [23], Z.S. Aliyev [1], Z.S. Aliyev and N.A. Mustafayeva [7], R. Ma and G. Dai [20] and others. In the case of bifurcation from zero, they in the works [1, 2, 9, 20, 21, 23] proved the existence of two families of unbounded continua, branching from points and intervals of a line of trivial solutions and contained in classes of functions that have the usual nodal properties. In the case of bifurcation from infinity, works [7, 20, 22-25] show the existence of two families of global continua bifurcating from points and intervals of the line $R \times \{\infty\}$ and contained in classes of functions which have the usual nodal properties in the neighborhood of these points and intervals. Note that in the works [2, 5, 6, 10] these results were also established for nonlinear Sturm-Liouville problems of second and fourth orders with a definite weight function and a spectral parameter in the boundary condition. The global bifurcation from zero and infinity of solutions to nonlinear Sturm-Liouville problems of second and fourth orders with indefinite weight functions has been intensively studied recently in [3, 4, 8, 9]. In these works, the authors show the existence of four families of global continua of solutions that have the above properties.

The purpose of this paper is to simultaneously study the behavior and structure of global continua of solutions to problem (1)-(3) branching from zero and infinity.

2. Preliminary

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x) y, \; x \in (0, 1), \\ b_0 y(0) = d_0 p(0) y'(0), \\ (a_1 \lambda + b_1) y(1) = d_1 p(1) y'(1) \end{cases} \tag{6}$$

which obtained from (1)-(3) by setting $f \equiv 0$. By [11, Theorem 3.2] the eigenvalues of the linear spectral problem (6) are real, simple and form two infinitely increasing and infinitely decreasing sequences

$$0 < \lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots \quad \text{and} \quad 0 > \lambda_1^- > \lambda_2^- < \ldots > \lambda_k^- > \ldots ,$$

respectively. Moreover, for each $k \in \mathrm{N}$ the eigenfunctions $y_k^+(x)$ and $y_k^-(x)$ corresponding to the eigenvalues λ_k^+ and λ_k^-, respectively, have exactly $k-1$ simple zeros in the interval $(0, 1)$.

As is known, the oscillatory properties of eigenfunctions of linear problems play an important role in the study of global bifurcation of solutions to nonlinear eigenvalue problems. According to this, we present some important classes of functions constructed in the papers [15-17].

By $(b.c.)_0$ and $(b.c.)_\lambda$ we denote the sets of functions satisfying the boundary conditions (2) and (3), respectively.

Let $E = C^1[0, 1] \bigcap (b.c.)_0$ be the Banach space with the usual norm

$$\| u \|_1 = \| u \|_\infty + \| u' \|_\infty ,$$

where $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$.

From now on σ and ν will denote either $+$ or $-$; $-\sigma$ and $-\nu$ will denote the opposite sign to σ and ν.

For each fixed $\lambda \in R$, each $k \in \mathrm{N}$, each σ and each ν let $S_{k,\lambda}^{\sigma,\nu}$ be the set of functions $u \in E$ which satisfy the following conditions:

(i) $u \in (b.c.)_\lambda$;

(ii) the function $u(x)$ has exactly $k-1$ simple zeros in the interval $(0,1)$;

(iii) $\sigma \left\{ \int_0^1 r(x) u^2(x) dx + \frac{a_1}{d_1} u^2(1) \right\} > 0$;

(iv) the function $\nu u(x)$ is positive in a deleted neighbourhood of the point $x = 0$.

Remark 1. It follows from the definition of the sets $S_{k,\lambda}^{\sigma,\nu}, k \in \mathrm{N}, \lambda \in R$, that these sets are open in E. Moreover, if $u \in \partial S_{k,\lambda}^{\sigma,\nu}$, then either

(i) there exists $\xi \in [0, 1]$ such that $u(\xi) = u'(\xi) = 0$, or

(ii) $\int_0^1 r(x) u^2(x) dx + \frac{a_1}{d_1} u^2(1) = 0$.

For each $k \in \mathrm{N}$, each σ and each ν we define the set $S_k^{\sigma,\nu}$ by

$$S_k^{\sigma,\nu} = \bigcup_{\lambda \in R} S_{k,\lambda}^{\sigma,\nu}.$$

Note that for each $k \in \mathrm{N}$, each σ and each ν the set $S_k^{\sigma,\nu}$ is open in E. Moreover, if $u \in \partial S_k^{\sigma,\nu}$, then by Remark 1 either (i) there exists $\xi \in [0, 1]$ such that $u(\xi) = u'(\xi) = 0$, or (ii) $\int_0^1 r(x) u^2(x) dx + \frac{a_1}{d_1} u^2(1) = 0$.

Lemma 1. If (λ, y) is a nontrivial solution of problem (1)-(3) such that $y \in \partial S_k^{\sigma,\nu}$, then $y \equiv 0$.

Proof. Let (λ, y) be a nontrivial solution of problem (1)-(3) such that $y \in \partial S_k^{\sigma,\nu}$. Then either (i) there exists $\xi \in [0, 1]$ such that $y(\xi) = y'(\xi) = 0$, or (ii) $\int_0^1 r(x) y^2(x) dx + \frac{a_1}{d_1} y^2(1) = 0$.

Multiplying both sides of equation (1) by y, integrating the resulting relation in the range from 0 to 1, applying the formula of integration by parts and taking into account the boundary conditions (2) and (3) we obtain

$$\int_0^1 \left\{p(x)y'^2(x) + q(x)y^2(x)\right\}dx + N[y] = \lambda\left[\int_0^1 r(x)y^2(x)dx + \frac{a_1}{d_1}y^2(1)\right] + \int_0^1 y(x)f(x, y(x), y'(x), \lambda)dx, \tag{7}$$

where

$$N[y] = \frac{b_0}{d_0} \text{ for } d_0 \neq 0,\ N[y] = 0 \text{ for } d_0 = 0. \tag{8}$$

If $\int_0^1 r(x)\, y^2(x)dx + \frac{a_1}{d_1}y^2(1) = 0,$ then it follows from (7) that

$$\int_0^1 \left\{p(x)y'^2(x) + q(x)y^2(x)\right\}dx + N[y] = \int_0^1 y(x)f(x, y(x), y'(x), \lambda)dx. \tag{9}$$

By (8), the left-hand side of (9) is positive, and by (4), the right-hand side of (9) is non-positive, which leads to a contradiction.

If there exists $\xi \in [0, 1]$ such that $u(\xi) = u'(\xi) = 0,$ then the proof of this lemma is similar to that of [20, Lemma 2.2]. The proof of Lemma 2.2 is complete.

3. Operator interpretation of problem (1)-(3) and some necessary results

Let $H = L_2(0, 1) \oplus C$ be a Hilbert space with the scalar product

$$(\hat{y}, \hat{\vartheta}) = \int_0^1 y(x)\overline{\vartheta(x)}\,dx + |a_1|^{-1}\, m\bar{s},$$

where

$$\hat{y} = \begin{pmatrix} y \\ m \end{pmatrix} \in H, \quad \hat{\vartheta} = \begin{pmatrix} \vartheta \\ s \end{pmatrix} \in H.$$

We define in the space H an operator

$$A\hat{y} = A\begin{pmatrix} y \\ m \end{pmatrix} = \begin{pmatrix} \ell(y) \\ d_1 p(1)y'(1) - b_1 y(1) \end{pmatrix}$$

with the domain

$$D(A)=\left\{\hat{y}=\begin{pmatrix} y \\ m \end{pmatrix}\in H: y, y'\in AC[0,1],\, \ell(y)\in L_2(0,1),\right.$$

$$\left. b_0 y(0)=d_0 p(0) y'(0),\, m=a_1 y(1)\right\}$$

which is dense everywhere in H. Moreover, we define the operators $B: H\to H$ and $F: R\times D(A)\to H$ as follows:

$$B\hat{y}=B\begin{pmatrix} y \\ m \end{pmatrix}=\begin{pmatrix} ry \\ m \end{pmatrix},$$

$$F(\lambda,\hat{y})=F\left(\lambda, \begin{pmatrix} y \\ m \end{pmatrix}\right)=\begin{pmatrix} f(x,y,y',\lambda) \\ 0 \end{pmatrix}.$$

It is obvious that operators A, B and F are well defined. Direct verification shows that problem (1)-(3) is equivalent to the following nonlinear eigenvalue problem

$$A\hat{y}=\lambda B\hat{y}+F(\lambda,\hat{y}),\ y\in D(A). \tag{7}$$

Since $a_1 d_1>0$ and $b_1\le 0$ it follows from [16, Lemma 2.1] that A is a self-adjoint positive definite operator on $D(A)$. Consequently, the smallest eigenvalue λ_1 of this operator is positive.

We introduce the following notations:

$$I_k^+=[\lambda_k^+,\lambda_k^++d_k^+],\ I_k^-=[\lambda_k^--d_k^-,\lambda_k^-],\ k\in \mathrm{N},$$

where

$$d_k^+=\frac{M\lambda_k^+}{\lambda_1},\ d_k^-=-\frac{M\lambda_k^-}{\lambda_1}.$$

By the condition (5) there exist a positive number M and a positive sufficiently small number c_0 such that

$$\left|\frac{f(x,u,s,\lambda}{u}\right|\le M,\ x\in[0,1],\ (u,s)\in R^2, u\ne 0, \tag{5_1}$$
$$|u|+|s|\le c_0, |\lambda\in R.$$

Remark 2. Since conditions (4) and (5_1) are satisfied, it follows from Corollary 3.1 of [15] that for each $k\in \mathrm{N}$, each σ and each ν the set of bifurcation points of problem (1)-(3) with respect to the set $R\times S_k^{\sigma,\nu}$ is nonempty.

Moreover, if $(\lambda,0)$ is a bifurcation point of the nonlinear eigenvalue problem (1)-(3) with respect to the set $R\times S_k^{\sigma,\nu}$, then $\lambda\in I_k^\sigma$.

Let D the set of nontrivial solutions of problem (1)-(3).

For each $k\in\mathrm{N}$, each σ and each ν let $D_k^{\sigma,\nu}$ be the union of all the components $D_{k,\lambda}^{\sigma,\nu}$ of the closure of the set D, branching from bifurcation points $(\lambda,0)\in I_k^\sigma\times\{0\}$ with respect to the set $R\times S_k^{\sigma,\nu}$. By Remark 2 the set $\tilde{D}_k^{\sigma,\nu}$ is nonempty. Let $\tilde{D}_k^{\sigma,\nu}=D_k^{\sigma,\nu}\bigcup(I_k^\sigma\times\{0\})$. Then the set $\tilde{D}_k^{\sigma,\nu}$ is connected in $R\times E$, but the set $D_k^{\sigma,\nu}$ may not be connected in $R\times E$.

Theorem 1 [2, Theorem 3.2]. For each $k\in\mathrm{N}$, each σ and each ν the set $D_k^{\sigma,\nu}$ is unbounded in $R\times E$ and contained in $R^\sigma\times S_k^{\sigma,\nu}$.

By the condition (5) there exist a positive number M and a positive sufficiently large number c_1 such that

$$\left|\frac{f(x,u,s,\lambda}{u}\right|\le M,\ \ x\in[0,\,1],\,(u,s)\in R^2,u\ne 0,\ |u|+|s|\ge c_1,\lambda\in R. \qquad (5_2)$$

Remark 3. Since conditions (4) and (5_2) are satisfied, using operator interpretation (7) of problem (1)-(3) by following the arguments in [17] and in Section 5 of [7] we can show that for each $k\in\mathrm{N}$, each σ and each ν the set of asymptotic bifurcation points of problem (1)-(3) with respect to the set $R^\sigma\times S_k^{\sigma,\nu}$ is nonempty. Moreover, if (λ,∞) is a asymptotic bifurcation point of problem (1) – (3) with respect to the set $R^\sigma\times S_k^{\sigma,\nu}$, then $\lambda\in I_k^\sigma$.

For each $k\in\mathrm{N}$, each σ and each ν let $\hat{D}_k^{\sigma,\nu}$ be the union of all the components $\tilde{D}_{k,\lambda}^{\sigma,\nu}$ of the set D, branching from asymptotic bifurcation points $(\lambda,\infty)\in I_k^\sigma\times\{\infty\}$ with respect to the set $R^\sigma\times S_k^{\sigma,\nu}$ (Adding the points (λ,∞) to $R\times E$ and defining an appropriate topology on the resulting set, we obtain that (λ,∞) is an element of $R\times E$). Let $\tilde{\hat{D}}_k^{\sigma,\nu}=\hat{D}_k^{\sigma,\nu}\bigcup(I_k^\sigma\times\{\infty\})$. Then the set $\tilde{\hat{D}}_k^{\sigma,\nu}$ is connected in $R\times E$, but the set $\hat{D}_k^{\sigma,\nu}$ may not be connected in $R\times E$.

By following the arguments in Theorem 5.9 of [7] we can prove the following theorem.

Theorem 2. For each $k \in \mathrm{N}$, each σ and each ν the set $\hat{D}_k^{\sigma,\nu}$ is nonempty and at least one of the following holds:

(i) $\hat{D}_k^{\sigma,\nu}$ meets $I_{k'}^{\sigma} \times \{\infty\}$ with respect to $R^{\sigma} \times S_{k'}^{\sigma,\nu'}$ for some $(k',\nu') \neq (k,\nu)$;

(ii) $\hat{D}_k^{\sigma,\nu}$ meets $(\lambda,0)$ for some $\lambda \in R$;

(iii) the natural projection $P_R(\hat{D}_k^{\sigma,\nu})$ of $\hat{D}_k^{\sigma,\nu}$ onto $R \times \{0\}$ is unbounded.

In addition, if the union $\hat{D}_k^{\sigma} = \hat{D}_k^{\sigma,+} \bigcup \hat{D}_k^{\sigma,-}$ does not satisfy (ii) or (iii), then it must satisfy (i) with $k' \neq k$.

4. Global bifurcation from zero and infinity of nontrivial solutions of problem (1)-(3)

Theorem 3. For each $k \in \mathrm{N}$, each σ and each ν the following relation holds: $\hat{D}_k^{\sigma,\nu} \subset R^{\sigma} \times S_k^{\sigma,\nu}$, and consequently, alternative (i) of Theorem 2 cannot hold. Moreover, if the set $D_k^{\sigma,\nu}$ meets (λ, ∞) for some $\lambda \in R$, then $\lambda \in I_k^{\sigma}$, and if the set $\hat{D}_k^{\sigma,\nu}$ meets $(\lambda, 0)$ for some $\lambda \in R$, then $\lambda \in I_k^{\sigma}$.

Proof. By Lemma 1 we have $D \bigcap (R \times \partial S_k^{\sigma,\nu}) = \varnothing$. Then it follows that the sets $D \bigcap (R \times S_k^{\sigma,\nu})$ and $D \setminus (R \times S_k^{\sigma,\nu})$ are mutually separated in $R \times E$ (see [26, Definition 26.4]. Thus, according to [26, Corollary 26.6], any connected component of the set D must be a subset of either the set $D \bigcap (R \times S_k^{\sigma,\nu})$ or the set $D \setminus (R \times S_k^{\sigma,\nu})$. Since $\hat{D}_k^{\sigma,\nu}$ is the union of all components of the set D which intersect the set $R \times S_k^{\sigma,\nu}$, each of these components must be a subset of the set $R \times S_k^{\sigma,\nu}$, and consequently, $\hat{D}_k^{\sigma,\nu} \subset R^{\sigma} \times S_k^{\sigma,\nu}$.

Let the set $D_k^{\sigma,\nu}$ meets (λ, ∞) for some $\lambda \in R$. It is obvious that $\lambda \in R^{\sigma}$. Since $D_k^{\sigma,\nu} \subset R^{\sigma} \times S_k^{\sigma,\nu}$ it follows that (λ, ∞) is an asymptotic bifurcation point of problem (1)-(3) with respect to the set $R^{\sigma} \times S_k^{\sigma,\nu}$. Then by Remark 3 we get $\lambda \in I_k^{\sigma}$.

Now let the set $\hat{D}_k^{\sigma,\nu}$ meets $(\lambda, 0)$ for some $\lambda \in R$. Obviously, $\lambda \in R^\sigma$. By Theorem 1 we have $D_k^{\sigma,\nu} \subset R^\sigma \times S_k^{\sigma,\nu}$, and consequently, $(\lambda, 0)$ is a bifurcation point of problem (1)-(3) with respect to the set $R^\sigma \times S_k^{\sigma,\nu}$. Then it follows from Remark 2 that $\lambda \in I_k^\sigma$. The proof of this theorem is complete.

Corollary 1. If for some $k \in \mathrm{N}$ alternative (iii) of Theorem 2 does not hold, then

$$D_k^{\sigma,\nu} = \hat{D}_k^{\sigma,\nu}.$$

Remark 4. If alternative (ii) of Theorem 2 does not hold, then the question naturally arises whether the set $D_k^{\sigma,\nu}$ intersects the set $\hat{D}_k^{\sigma,\nu}$. By following the arguments in Example 4.1 of [7] we can show that both cases are possible for these sets.

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