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ON THE FRAME PROPERTY OF THE WEIGHT SYSTEM OF EXPONENTS IN GENERALIZED GRAND LEBESGUE SPACES

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Abstract

The work is devoted to the study of the frame property of the weight system of exponentials in the generalized space of grand Lebesgue, with a weight function of a general form. The basicity of the system of exponents in the subspace of the generalized space of grand Lebesgue generated by the shift operator is established. Criteria for the K-Besselianness and K-frameness of the weight system of exponentials in the subspace of the generalized space of grand Lebesgue are proved. In this case, as the space K, we take the space of sequences from the coefficients of the expansion in terms of the system of exponentials in ihis the subspace. In particular, criteria for the K-Besselianness and K-frameness are studied. Note that these results are generalizations of the criteria for Besselianness and frameness of a weight system of exponentials with a power-law weight in the Hilbert space studied in [6].

Keywords: exponential weight system; basis; frame; generalized grand Lebesgue space; shift operator.

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1. Introduction

As is known, in the work of K.I.Babenko [1], the basicity of the system $\left\{|t|^{\alpha} e^{\mathrm{int}}\right\}_{n\in Z}$ in $L_2(-\pi,\pi)$ for $|\alpha|<\frac{1}{2}$ is established, which is a Riesz basis only in the case $\alpha=0$. This result served as an answer to the question posed by N.K.Bari in [2] about the existence of a normalized basis that is not a Riesz basis. The case of a basicity in the space $L_2(-\pi,\pi)$ of a weight system of exponentials with a weight function of a general form was considered by V.F. Gaposhkin in [3]. It follows from the results of [5, 6] that a necessary and sufficient condition for a system $\left\{\rho(t)e^{\mathrm{int}}\right\}_{n\in Z}$ to be a basis in $L_p(0,2\pi)$ is the Mackenhoupt condition $\rho(t)\in A_p$, i.e. fulfillment of the condition

$$\frac{1}{|I|}\int_I \rho(t)dt\left(\frac{1}{|I|}\int_I \rho(t)^{-\frac{1}{p-1}}dt\right)^{p-1}<+\infty\,,$$

for any interval $I\subset[-\pi,\pi]$. The frame properties of a weighted system of exponentials $\left\{\rho(t)e^{\mathrm{int}}\right\}_{n\in Z}$ and trigonometric systems of sines and cosines with a power-law weight in the space $L_p(-\pi,\pi)$ were studied in [6-8].

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It is established in [6] that the system $\left\{|t|^{\alpha} e^{\mathrm{int}}\right\}_{n\in Z}$ forms a frame in $L_2(0,2\pi)$ only in the case $\alpha=0$. Note that the concept of a frame in Hilbert spaces was introduced in 1952 by R.J.Duffin and A.C.Schaeffer in [8] in the study of nonharmonic Fourier series with respect to perturbed exponential systems. In the same paper, the concept of an abstract frame in Hilbert spaces is introduced as a system $\{f_n\}_{n\in N}$ of a Hilbert space H for which there exist constants $A>0$ and $B>0$ such that the condition

$$A\,\|f\|_H^2\le\sum_{n=1}^{\infty}|(f,f_n)|^2\le B\,\|f\|_H^2\,,$$

holds for every $f\in H$, where $\|\cdot\|_H$ is the norm in H, generated by the scalar product $(\cdot,\cdot)$. The advantage of a frame is that each element of the Hilbert space has a decomposition into frame elements. Frames have important applications in signaling processes, information compression and processing, etc. Frames in Banach spaces were first studied in [10, 11]. Banach generalizations of frames were also studied in [12-14].

The purpose of this article is to study the frame property of a weight system $\{\rho(t)e^{\mathrm{int}}\}_{n\in Z}$ in a separable subspace $G_{p),\theta}(0,2\pi)$ of the generalized grand Lebesgue space $L_{p),\theta}(0,2\pi)$, $p>1$, in which the shift operator converges to the identity operator. Criteria of Besselianness and frameness of a system $\{\rho(t)e^{\mathrm{int}}\}_{n\in Z}$ with respect to the space of sequences of expansion coefficients in terms of the basis $\{e^{\mathrm{int}}\}_{n\in Z}$ in spaces $G_{p),\theta}(0,2\pi)$, $p>1$, are proved. From the results obtained, in particular, we obtain criteria for Besselianness and frameness of a weight system $\{\rho(t)e^{\mathrm{int}}\}_{n\in Z}$ in Lebesgue spaces $L_p(0,2\pi)$, $p>1$. Note that the grand Lebesgue space was introduced in [15], and the generalized grand Lebesgue space in [16]. In connection with important applications in the theory of partial differential equations, interest in these spaces has greatly increased. Among the works devoted to research in these spaces, the works [17-21] can be noted.

2. Preliminary information

Here are some standard notations: Z is the set of integers; N is the set of natural numbers; Z_+ is the set of non-negative integers; $|E|$ is the Lebesgue measure of the set $E\subset R$; χ_E is the characteristic function of the set $E\subset R$.

Let $p>1$, $\theta\geq 0$, $L_{p),\theta}(0,2\pi)$ be a generalized grand Lebesgue space of functions $f(x)$ measurable on $(0,2\pi)$ such that

$$\|f\|_{p),\theta}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^{\theta}}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}<+\infty .$$

The space $L_{p),\theta}(0,2\pi)$ for $\theta = 0$ coincides with the Lebesgue space $L_p(0,2\pi)$, and for $\theta = 1$ with the grand Lebesgue space $L_{p)}(0,2\pi)$. There is (see [21]) the following embedding

$$L_p(0,2\pi) \subset L_{p),\theta}(0,2\pi) \subset L_{p-\varepsilon}(0,2\pi),\ 0 < \varepsilon < p-1.$$

Denote by $(L_{p),\theta}(0,2\pi))'$ the space associated to the space $L_{p),\theta}(0,2\pi)$ equipped with the norm

$$\|g\|_{p)',\theta} = \sup\left\{ \int_0^{2\pi} |f(x)g(x)|dx : f \in L_{p),\theta}(0,2\pi), \|f\|_{p),\theta} \le 1 \right\}.$$

Let $G_{p),\theta}(0,2\pi)$ be a closed subspace $L_{p),\theta}(0,2\pi)$ which is the closure of the linear envelope of functions $f \in L_{p),\theta}(0,2\pi)$ satisfying the condition $T_\delta f \to f$ at $\delta \to +0$, where T_δ is the shift operator, i.e. $T_\delta f(x) = f(x+\delta)$, $x+\delta \in [0,2\pi]$ and $T_\delta f(x) = 0$, $x+\delta \notin [0,2\pi]$. As in the case of the space $L_{p)}(0,2\pi)$ (see [22, 23]), it can be shown that the set $C_0^\infty[0,2\pi]$ is dense in $G_{p),\theta}(0,2\pi)$.

Let X be some Banach space, X^* its dual space, and K some Banach space of sequences of scalars.

The following definition gives the notion of a frame in Banach spaces.

Definition 1 ([13]). *The system $\{x_n^*\}_{n\in N} \subset X^*$ is called a frame for X with respect to K (K-frame in X) if the following conditions are true*

i) $\{x_n^*(x)\}_{n\in N} \in K$ for $\forall x \in X$;

ii) there exist constants $0 < A \le B < +\infty$ such that

$$A\|x\|_X \le \left\| \{x_n^*(x)\}_{n\in N} \right\|_K \le B\|x\|_X. \quad (1)$$

In this case, the numbers A and B are called the lower and upper boundaries of the frame, respectively. If at least i) and the upper condition in (1) are satisfied, $\{x_n^*\}_{n\in N}$ is called K-Bessel for X with boundary B.

We will also use the following basicity criterion.

Statement 1 ([24]). *A complete in* X *sequence* $\{x_n\}_{n\in N}$ *forms a basis for* X *if and only if there exists a constants* $M>0$ *such that*

$$\left\|\sum_{k=1}^{n} a_k x_k\right\|_X \le M\left\|\sum_{k=1}^{m} a_k x_k\right\|_X$$

whenever $a_1, a_2, \ldots, a_m$ *are arbitrary scalars and* $n \le m$.

3. On basicity and frame property of the exponential weight system

The following theorem proves the basicity of the system of exponents $\{e^{\mathrm{int}}\}_{n\in Z}$ in $G_{p),\theta}(0,2\pi)$.

Theorem 1. *The system of exponents* $\{e^{\mathrm{int}}\}_{n\in Z}$ *forms a basis in* $G_{p),\theta}(0,2\pi)$.

Proof. The validity of the statement for $\theta=0$ follows from the results of [25]. Let us show the validity of the statement for $\theta>0$. By virtue of the completeness of the system $\{e^{\mathrm{int}}\}_{n\in Z}$ in $L_p(0,2\pi)$ and the density of $L_p(0,2\pi)$ in $G_{p),\theta}(0,2\pi)$, we obtain the completeness of the system $\{e^{\mathrm{int}}\}_{n\in Z}$ in $G_{p),\theta}(0,2\pi)$. Then, in order to prove the theorem according to Statement 1, it suffices to find a constants $M>0$ such that

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p),\theta} \le M\left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p),\theta} \tag{2}$$

for any constants $a_{-(m+s)},\ldots,a_{n+r}\in C$ and $n,m,r,s\in Z_+$. Let $0<\varepsilon_0<p-1$ be an arbitrary fixed number. By virtue of the fact that $\{e^{\mathrm{int}}\}_{n\in Z}$ is a basis in $L_p(0,2\pi)$ and $L_{p-\varepsilon_0}(0,2\pi)$, there are constants $M_0, M_1>0$ such that for arbitrary constants $a_{-(m+s)},\ldots,a_{n+r}\in C$ and $n,m,r,s\in Z_+$ we have

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p} \le M_0\left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p}, \tag{3}$$

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon_0} \leq M_1 \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon_0} . \tag{4}$$

Then using Hölder's inequality, taking into account (4) for any ε : $0<\varepsilon_0 \leq \varepsilon < p-1$ we get

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon} \leq M_2 \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon_0} , \tag{5}$$

where $M_2 = M_1 (2\pi)^{1-\frac{1}{p-\varepsilon_0}}$. By virtue of the Riesz-Thorin interpolation theorem, it follows from (3) and (4) that there is a number $M_3 > 0$ such that for any ε : $0<\varepsilon \leq \varepsilon_0$ we obtain

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon} \leq M_3 \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon} . \tag{6}$$

Therefore, taking into account (4)-(6) we get

$$\left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon} \leq$$

$$\leq \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon} + \sup_{\varepsilon_0\leq\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon} \leq$$

$$\leq M_3 \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon} + M_2 \sup_{\varepsilon_0\leq\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-m}^{n} a_k e^{ikt}\right\|_{p-\varepsilon_0} \leq$$

$$\leq M_3 \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon} + M_2 \varepsilon_0^{-\frac{\theta}{p-\varepsilon_0}} \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \varepsilon_0^{\frac{\theta}{p-\varepsilon_0}} \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p-\varepsilon_0} \leq$$

$$\leq M_3 \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p),\theta} + M_4 \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p),\theta} = M \left\|\sum_{k=-(m+s)}^{n+r} a_k e^{ikt}\right\|_{p),\theta} ,$$

where $M_4 = M_2 \varepsilon_0^{-\frac{\theta}{p-\varepsilon_0}} \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}}$. Thus (2) is satisfied, and hence the system $\{e^{\text{int}}\}_{n\in Z}$ forms a basis in $G_{p),\theta}(0,2\pi)$. The theorem is proved.

As in the case of $\theta = 1$ [26], it is shown that the system of functionals

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\text{int}}dt\,,\ f \in G_{p),\theta}(0,2\pi)\,,$$

forms in $G_{p),\theta}(0,2\pi)$ a biorthogonal system to system $\{e^{\text{int}}\}_{n\in Z}$. Denote by $K_{p),\theta}$ the space of sequences from the coefficients of the expansions in terms of the basis $\{e^{\text{int}}\}_{n\in Z}$ in $G_{p),\theta}(0,2\pi)$, i.e.

$$K_{p),\theta} = \left\{ \{a_n\}_{n\in Z} \subset C : \sum_{n\in Z} a_n e^{\text{int}} \in G_{p),\theta}(0,2\pi) \right\}.$$

Let's put $K_{p),0} = K_p$ and $K_{p),1} = K_{p)}$. The space $K_{p),\theta}$ with coordinate-wise linear operations becomes a Banach space with respect to the norm

$$\left\| \{a_n\}_{n\in Z} \right\|_{K_{p),\theta}} = \sup_{m,n\in Z_+} \left\| \sum_{k=-m}^{n} a_k e^{ikt} \right\|_{p),\theta}.$$

Therefore, the operator $F: K_{p),\theta} \to G_{p),\theta}(0,2\pi)$ given by formula $F(\{a_n\}_{n\in Z}) = \sum_{n\in Z} a_n e^{\text{int}}$ is isomorphic, and thus, there are numbers $a, b > 0$ such that

$$a\|f\|_{p),\theta} \le \left\| \{a_n(f)\}_{n\in Z} \right\|_{K_{p),\theta}} \le b\|f\|_{p),\theta},\ f \in G_{p),\theta}(0,2\pi)\,. \tag{7}$$

Let us study the $K_{p),\theta}$-frame property of the following weight system of exponents of the form

$$\{\mu(t)e^{\text{int}}\}_{n\in Z}\,, \tag{8}$$

where $\mu(t)$ is some measurable function on $(0,2\pi)$ and nonzero a.e.

Let us first study the $K_{p),\theta}$ - Besselianness of system (8).

Theorem 2. *Let* $\mu \in (G_{p),\theta}(0,2\pi))'$. *Then the system* $\{\mu(t)e^{\text{int}}\}_{n\in Z}$ *is* $K_{p),\theta}$

-Bessel for $G_{p),\theta}(0,2\pi)$ if and only if there exists a constant $c>0$ such that the relation

$$|\mu(t)| \le c \tag{9}$$

holds almost everywhere on $[0,2\pi]$.

Proof. Let system (8) be $K_{p),\theta}$-Bessel in $G_{p),\theta}(0,2\pi)$ with boundary $B>0$, i.e. there is a ratio

$$\left\| \{ \nu_n(f) \}_{n\in Z} \right\|_{K_{p),\theta}} \le B \|f\|_{p),\theta}, \; f \in G_{p),\theta}(0,2\pi), \tag{10}$$

where $\nu_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mu(t)e^{-\mathrm{int}}dt$. Take an arbitrary $f \in G_{p),\theta}(0,2\pi)$. Let

$$g = \sum_{n\in Z} \nu_n(f) e^{\mathrm{int}}.$$

It is clear that

$$a_n(g) = \nu_n(f), \; n \in Z.$$

Hence $g = \mu f$. Then, taking into account (7) and (10), we get

$$a\|g\|_{p),\theta} \le \left\| \{ a_n(g) \}_{n\in Z} \right\|_{K_{p),\theta}} = \left\| \{ \nu_n(f) \}_{n\in Z} \right\|_{K_{p),\theta}} \le B \|f\|_{p),\theta}.$$

So, we have

$$\|\mu f\|_{p),\theta} \le \frac{B}{a}\|f\|_{p),\theta}, \; \forall f \in G_{p),\theta}(0,2\pi). \tag{11}$$

From (11) it follows that

$$|\mu(t)| \le \frac{B}{a}, \text{ a.e. } t \in [0,2\pi]. \tag{12}$$

In fact, let $c > \frac{B}{a}$ and $E_c = \{ t : |\mu(t)| \ge c \}$. Using the right side of inequality (11) for $f = \chi_{E_c}$, we obtain

$$\frac{B}{a}\|\chi_{E_c}\|_{p),\theta} \ge \|\mu\chi_{E_c}\|_{p),\theta} \ge c\|\chi_{E_c}\|_{p),\theta}.$$

If $|E_c| \ne 0$, then

$$\left\|\chi_{E_c}\right\|_{p),\theta} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^{\theta}}{2\pi}\left|E_c\right|\right)^{\frac{1}{p-\varepsilon}} \geq \frac{(p-1)^{\theta}}{2\pi}\left|E_c\right|,$$

and it follows from the last inequality that $c \leq \frac{B}{a}$. This contradicts the condition $c > \frac{B}{a}$. So $\left|E_c\right| = 0$. Hence it follows that

$$\left|\left\{t : \left|\mu(t)\right| \geq \frac{B}{a} + \frac{1}{n}\right\}\right| = 0\,,\ \forall n \in N\,.$$

On the other hand, it is obvious that

$$\left\{t : \left|\mu(t)\right| > \frac{B}{a}\right\} = \bigcup_{n=1}^{\infty}\left\{t : \left|\mu(t)\right| \geq \frac{B}{a} + \frac{1}{n}\right\}.$$

Therefore, we have $\left|\left\{t : \left|\mu(t)\right| > \frac{B}{a}\right\}\right| = 0$, i.e. for a.e. $t \in [0,2\pi]$ (12) is satisfied.

Conversely, let there be a number $c > 0$ such that (9) holds almost everywhere on $[0,2\pi]$. Then it is obvious that for $\forall f \in G_{p),\theta}(0,2\pi)$ the inclusion $g = \mu f \in G_{p),\theta}(0,2\pi)$ and the inequality

$$\left\|\mu f\right\|_{p),\theta} \leq c\left\|f\right\|_{p),\theta}. \tag{13}$$

are true. Therefore, taking into account the equality $a_n(g) = \nu_n(f)$, $n \in Z$, using (7) and (13) we obtain

$$\left\|\left\{\nu_n(f)\right\}_{n\in Z}\right\|_{K_{p),\theta}} = \left\|\left\{a_n(g)\right\}_{n\in Z}\right\|_{K_{p),\theta}} \leq b\left\|g\right\|_{p),\theta} =$$

$$= b\left\|\mu f\right\|_{p),\theta} \leq cb\left\|f\right\|_{p),\theta}.$$

Thus, for $\forall f \in G_{p),\theta}(0,2\pi)$ we have

$$\left\|\left\{\nu_n(f)\right\}_{n\in Z}\right\|_{K_{p),\theta}} \leq cb\left\|f\right\|_{p),\theta},$$

i.e. system (8) is $K_{p),\theta}$-Bessel in $G_{p),\theta}(0,2\pi)$. The theorem is proved.

In the following theorem, the $K_{p),\theta}$ -frameness of system (8) is studied.

Theorem 3. *Let* $\mu \in (G_{p),\theta}(0,2\pi))'$. *Then the system* $\{\mu(t)e^{\mathrm{int}}\}_{n\in Z}$ *forms a* $K_{p),\theta}$ *-frame for* $G_{p),\theta}(0,2\pi)$ *if and only if there exist numbers* $c, d > 0$ *such that the relation*

$$d \le |\mu(t)| \le c \tag{14}$$

holds almost everywhere on $[0,2\pi]$.

Proof. Let system (8) form a $K_{p),\theta}$ -frame for $G_{p),\theta}(0,2\pi)$ with boundaries $A > 0$ and $B > 0$. Then for any $f \in G_{p),\theta}(0,2\pi)$ the relation

$$A\|f\|_{p),\theta} \le \left\|\{\nu_n(f)\}_{n\in Z}\right\|_{K_{p),\theta}} \le B\|f\|_{p),\theta} \tag{15}$$

is valid. Hence, by Theorem 2, there exists a number $c > 0$ such that almost everywhere on $[0,2\pi]$ the relation

$$|\mu(t)| \le c\,.$$

is true. Let $f \in G_{p),\theta}(0,2\pi)$. It is clear that if

$$g = \sum_{n\in Z} \nu_n(f)e^{\mathrm{int}},$$

then $g = \mu f$ and $a_n(g) = \nu_n(f)$, $n \in Z$. Then, taking into account (7) and (15), we obtain

$$A\|f\|_{p),\theta} \le \left\|\{\nu_n(f)\}_{n\in Z}\right\|_{K_{p),\theta}} = \left\|\{a_n(g)\}_{n\in Z}\right\|_{K_{p),\theta}} \le b\|g\|_{p),\theta}, \text{ i.e.}$$

$$\frac{A}{b}\|f\|_{p),\theta} \le \|\mu f\|_{p),\theta}\,. \tag{16}$$

From (16) it follows that

$$\frac{A}{b} \le |\mu(t)|, \text{ a.e. } t \in [0,2\pi]. \tag{17}$$

Indeed, let $0 < c < \frac{A}{b}$ and $E_c = \{t : |\mu(t)| \le c\}$. Then for $f = \chi_{E_c}$ according to (15) and (16) we obtain

$$\frac{A}{b}\left\|\chi_{E_c}\right\|_{p),\theta} \le \left\|\mu\chi_{E_c}\right\|_{p),\theta} \le c\left\|\chi_{E_c}\right\|_{p),\theta}.$$

Therefore, if $\left|E_c\right| \ne 0$, then. The resulting contradiction proves $\left|E_c\right| = 0$, i.e. for a.e. (17) is fulfilled. If $\left|E_c\right| \ne 0$, then

$$\left\|\chi_{E_c}\right\|_{p),\theta} \ge r > 0,$$

and it follows from the last inequality that $\frac{A}{b} \le c$. This contradicts the condition $c < \frac{A}{b}$. So $\left|E_c\right| = 0$. In particular,

$$\left|\left\{t : \left|\mu(t)\right| \le \frac{A}{b} - \frac{1}{n}\right\}\right| = 0, \ \forall n \in N.$$

Then, by virtue of the equality

$$\left\{t : \left|\mu(t)\right| < \frac{A}{b}\right\} = \bigcup_{n=1}^{\infty}\left\{t : \left|\mu(t)\right| \le \frac{A}{b} - \frac{1}{n}\right\},$$

we get $\left|\left\{t : \left|\mu(t)\right| < \frac{A}{b}\right\}\right| = 0$, i.e. for a.e. $t \in [0,2\pi]$ (17) is satisfied.

Conversely, let there be numbers $c, d > 0$ such that (9) holds almost everywhere on $[0,2\pi]$. By Theorem 2, system (8) is $K_{p),\theta}$-Bessel in $G_{p),\theta}(0,2\pi)$ and

$$\left\|\left\{\nu_n(f)\right\}_{n\in Z}\right\|_{K_{p),\theta}} \le cb\left\|f\right\|_{p),\theta}, \forall f \in G_{p),\theta}(0,2\pi).$$

It remains to prove the left side of (12). Let's take $\forall f \in G_{p),\theta}(0,2\pi)$. Using (9), we get

$$d\left\|f\right\|_{p),\theta} \le \left\|\mu f\right\|_{p),\theta} \le c\left\|f\right\|_{p),\theta}. \qquad (18)$$

Obviously $g = \mu f \in G_{p),\theta}(0,2\pi)$. Therefore, taking into account $a_n(g) = \nu_n(f)$, $n \in Z$, using (7) and (18) we obtain

$$\left\|\{v_n(f)\}_{n\in Z}\right\|_{K_{p),\theta}} = \left\|\{a_n(g)\}_{n\in Z}\right\|_{K_{p),\theta}} \geq a\|g\|_{p),\theta} =$$

$$= a\|\mu f\|_{p),\theta} \geq da\|f\|_{p),\theta}.$$

Thus, for $\forall f \in G_{p),\theta}(0,2\pi)$

$$da\|f\|_{p),\theta} \leq \left\|\{v_n(f)\}_{n\in Z}\right\|_{K_{p),\theta}} \leq cb\|f\|_{p),\theta},$$

it is true, i.e. system (8) forms a $K_{p),\theta}$-frame for $G_{p),\theta}(0,2\pi)$. The theorem is proved.

From the proved theorem, in particular, it follows

Corollary 1. *Let* $\mu(t) = \prod_{k=0}^{r} |t - t_k|^{\alpha_k}$, $\alpha_k \in R$, $0 = t_0 < t_1 < ... < t_r = 2\pi$. *Then system* $\{\mu(t)e^{\text{int}}\}_{n\in Z}$ *forms a frame for* $G_{p)}(0,2\pi)$ *with respect to* $K_{p)}$ *if and only if* $\alpha_k = 0$, $k = \overline{0,r}$ *is true.*

Corollary 2. *Let* $\mu(t) = \prod_{k=0}^{r} |t - t_k|^{\alpha_k}$, $\alpha_k \in R$, $0 = t_0 < t_1 < ... < t_r = 2\pi$. *Then system* $\{\mu(t)e^{\text{int}}\}_{n\in Z}$ *forms a frame for* $L_p(0,2\pi)$ *with respect to* K_p *if and only if* $\alpha_k = 0$, $k = \overline{0,r}$ *is true.*

Remark 1. Note that this result in the case of the space $L_2(0,2\pi)$ was obtained in [6].

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