

SOLVABILITY OF SPECTRAL PROBLEM FOR A 3-DIMENSIONAL ELLIPTIC TYPE EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS

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Received 20 october 2023; accepted 19 january 2024

<https://doi.org/10.30546/209501.101.2025.2.1.03>

Abstract

A spectral boundary value problem for a 3-dimensional elliptic type equation with nonlocal boundary conditions is considered. The derived necessary conditions of solvability have singularities of a non-general character which are regularized according to the original scheme. The Fredholm property of the stated problem is proved.

Keywords: spectral problem, elliptic type equation, nonlocal boundary conditions, basic relationships, necessary conditions, solvability, regularization, Fredholm property.

Mathematics Subject Classification (2020): **35M12**

1. Introduction

As known the Fredholm property of local boundary value problems (Dirichlet, Neumann, Poincare) for PDE is proved by reduction to an integral equation and

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method of successive approximations with reduces it to a Fredholm equation of second kind. As in the theory of PDE usually equations of even order are considered this scheme is conventional in this case.

In the nonlocal boundary value problems for PDE the whole boundary is the carrier of the nonlocal boundary condition as if it “stitch together” the values of the unknown function and its derivatives on different parts of the boundary. The introduction of nonlocal conditions also eliminated the misunderstanding between the o.d.e. and partial differential equations, when the number of boundary conditions doesn’t coincide with the order of the equation.

In contrast to classical problems, for the nonlocal BVP there arose possibility of Fredholm property proof both for even and odd orders [7]-[10] by a principally new method: based on a fundamental solution of the principal part of the PDE and by means of the 2-nd Green’s formula so called “necessary conditions” of solvability are obtained. The Fredholm property is investigated just from the necessary conditions.

It should be noted that for linear ordinary differential equations these necessary conditions were obtained by A.A. Dezin in the form of usual boundary conditions [2]-[3]. For partial differential equations for the first time these conditions were considered and obtained by A.V.Bitsadze in the form of singular integral equations for two-dimensional Laplace equation [1]. The regularization was carried out using the method of successive approximations. A.A.Bitsadze called them both “necessary and sufficient conditions of solvability” [1].

In three-dimansional case the singularities have a special form and are not regularized as in the general case as we arrive at a Fredholm equation of the first kind which is a dead end. A new approach [7], [10], [4]-[6] is used to regularize the obtained singularities what allows to prove the Fredholmness of the posed problem.

2. Problem statement

Consider an elliptic type equation in a three-dimensional domain: $D \subset R^3$:

$$Lu = \Delta u(x) + \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} + a(x)u(x) = \lambda^2 u(x), \quad x \in D \subset R^3, \quad (1)$$

with nonlocal boundary conditions on the Lyapunov boundary:

$$l_i u = \sum_{k=1}^2 \sum_{j=1}^3 \alpha_{ij}^{(k)}(x') \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_k(x')} + \sum_{k=1}^2 \alpha_i^{(k)}(x') u(x) \Big|_{x_3=\gamma_k(x')} = \varphi_i(x'),$$

$$i = 1, 2; \quad x' = (x_1, x_2) \in S, \quad (2)$$

$$u(x) = f_0(x), \quad x \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2, \quad (3)$$

where $S = \text{proj}_{Ox_1x_2} D$, Γ_1 и Γ_2 are upper and lower halvesurfaces of the boundary Γ respectively:

$$\Gamma_k = \{ \xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'), \xi' = (\xi_1, \xi_2) \in S \},$$

$\xi_3 = \gamma_k(\xi_1, \xi_2)$, $k = 1, 2$, are equations of Γ_1 and Γ_2 ; the functions $\gamma_k(\xi')$, $k = 1, 2$, are twice differentiable with respect to both of the variables ξ_1, ξ_2 ; the coefficients $\alpha_{ij}^{(k)}(x')$, $\alpha_i^{(k)}(x')$, $\varphi_i(x')$ are continuous functions.

The fundamental solution to equation (1) is the same as for the three-dimensional Helmholtz equation : $L_0 u = \Delta u(x) + k^2 u(x)$, when $k = i\lambda$ [11]:

$$U(x - \xi) = -\frac{1}{4\pi} \frac{e^{-\lambda|x-\xi|}}{|x-\xi|}. \quad (4)$$

3. Basic relationships and necessary conditions

Multiplying equation (1) by the fundamental solution (4), integrating it over the domain D by parts

$$\begin{aligned} \int_D Lu(x)U(x-\xi)dx &= \int_D u(x)L_0U(x-\xi)dx + \\ &+ \sum_{j=1}^3 \int_{\Gamma} \left[\left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(\nu, x_j) dx \right] + \\ &+ \sum_{k=1}^3 \int_D a_k(x) \frac{\partial u(x)}{\partial x_k} U(x-\xi) dx + \int_D a(x) u(x) U(x-\xi) dx \end{aligned}$$

and taking into account that $L_{0,x} U(x-\xi) = \delta(x-\xi)$ where $\delta(x-\xi)$ is the Dirac δ -function we'll get **the first basic relationship**:

$$\begin{aligned}
 & - \sum_{j=1}^3 \int_{\Gamma} \left[\left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(v, x_j) dx \right] - \\
 & - \sum_{k=1}^3 \int_D a_k(x) \frac{\partial u(x)}{\partial x_k} U(x-\xi) dx - \int_D a(x) u(x) U(x-\xi) dx \\
 & = \int_D u(x) \delta(x-\xi) dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2} u(\xi), & \xi \in \Gamma. \end{cases} \quad (5)
 \end{aligned}$$

Here the first relationship gives the representation of the general solution of equation (1) and the second expression in (5) is the **first necessary condition**.

Taking into account $\frac{\partial U(x-\xi)}{\partial x_j} = \frac{e^{-\lambda|x-\xi|}(\lambda|x-\xi|+1)\cos(x-\xi, x_j)}{4\pi|x-\xi|^2}$ we obtain

the **1st necessary condition** ($\xi \in \Gamma$) in the form

$$\begin{aligned}
 \frac{1}{2} u(\xi) = & - \int_{\Gamma} \frac{\partial u(x)}{\partial v} U(x-\xi) dx - \int_{\Gamma} u(x) \frac{e^{-\lambda|x-\xi|}(\lambda|x-\xi|+1)\cos(x-\xi, v_x)}{4\pi|x-\xi|^2} dx - \\
 & - \sum_{k=1}^3 \int_D a_k(x) \frac{\partial u(x)}{\partial x_k} U(x-\xi) dx - \int_D a(x) u(x) U(x-\xi) dx \quad (6)
 \end{aligned}$$

where all the integrands a weak singularity as the order of singularity doesn't exceed the multiplicity of the integrals.

Thus we have proved

Theorem 1. *Let a convex along the direction Ox_3 domain $D \subset R^3$ be bounded with the boundary Γ which is a Lyapunov surface. Then the obtained first necessary condition (6) is regular.*

If we disclose two first surface integrals in (6) over the lower and the upper half surfaces Γ_k , $k = 1, 2$, we get:

$$\frac{1}{2} u(\xi) \Big|_{\xi_3 = \gamma_k(\xi)} =$$

$$\begin{aligned}
 & - \sum_{i=1}^2 \int_{\Gamma_i} \frac{\partial u(x)}{\partial \nu} U(x-\xi) dx - \sum_{i=1}^2 \int_{\Gamma_i} u(x) \frac{e^{-\lambda|x-\xi|} (\lambda|x-\xi|+1) \cos(x-\xi, \nu_x)}{4\pi|x-\xi|^2} dx - \\
 & - \sum_{m=1}^3 \int_D a_m(x) \frac{\partial u(x)}{\partial x_m} U(x-\xi) dx - \int_D a(x) u(x) U(x-\xi) dx, \quad \xi \in \Gamma_k, k=1, 2. \quad (7)
 \end{aligned}$$

Multiplying (1) by $\frac{\partial U(x-\xi)}{\partial x_i}, i=\overline{1,3}$, , integrating it over the domain D we

obtain the rest of **three basic relationships**:

$$\begin{aligned}
 \int_D L u \frac{\partial U(x-\xi)}{\partial x_i} dx &= \int_D L_0 u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx + \int_D \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} \frac{\partial U(x-\xi)}{\partial x_i} dx + \\
 &+ \int_D a(x) u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx,
 \end{aligned}$$

whence after integrating by parts we get **the second basic relationship**

$$\begin{aligned}
 & \int_{\Gamma} \frac{\partial u(x)}{\partial x_m} \left[\frac{\partial U(x-\xi)}{\partial x_m} \cos(\nu_x, x_i) - \frac{\partial U(x-\xi)}{\partial x_i} \cos(\nu_x, x_m) \right] dx + \\
 & + \int_{\Gamma} \frac{\partial u(x)}{\partial x_l} \left[\frac{\partial U(x-\xi)}{\partial x_l} \cos(\nu_x, x_i) - \frac{\partial U(x-\xi)}{\partial x_i} \cos(\nu_x, x_l) \right] dx + \\
 & - \int_{\Gamma} \frac{\partial u(x)}{\partial x_i} \frac{\partial U(x-\xi)}{\partial \nu_x} dx - \int_D \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} \frac{\partial U(x-\xi)}{\partial x_i} dx - \\
 & - \int_D a(x) u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in D, \\ \frac{1}{2} \frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in \Gamma, \end{cases} \quad i=\overline{1,3}, \quad (8)
 \end{aligned}$$

where the numbers i, m, l make a permutation of numbers $1,2,3$. The second expressions in (8) are **the second necessary conditions** ($\xi \in \Gamma, i=\overline{1,3}$).

Introducing the designations

$$K_{ij}(x, \xi) = e^{-\lambda|x-\xi|} (|x-\xi|\lambda+1) (\cos(x-\xi, x_i) \cos(\nu_x, x_j) - \cos(x-\xi, x_j) \cos(\nu_x, x_i))$$

we can rewrite **the second necessary conditions** in (8) in the form:

$$\begin{aligned} \frac{1}{2} \frac{\partial u(\xi)}{\partial \xi_i} = & \int_{\Gamma} \frac{\partial u(x)}{\partial x_m} \frac{K_{mi}(x, \xi)}{4\pi|x-\xi|^2} dx + \int_{\Gamma} \frac{\partial u(x)}{\partial x_l} \frac{K_{li}(x, \xi)}{4\pi|x-\xi|^2} dx . \\ & - \int_{\Gamma} \frac{\partial u(x)}{\partial x_i} \frac{\partial U(x-\xi)}{\partial v_x} dx - \int_D \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} \frac{\partial U(x-\xi)}{\partial x_i} dx - \\ & - \int_D a(x)u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx . \end{aligned} \quad (9)$$

where $i = \overline{1,3}$ and the numbers i, m, l make a permutation of numbers 1,2,3.

If we disclose two first surface integrals in (9) ($i = 1,2,3$) over the lower and the upper half surfaces Γ_k , $k = 1,2$:

$$\begin{aligned} \frac{\partial u}{\partial \xi_i} \Big|_{\xi_3=\gamma_k(\xi')} = & \sum_{j=1}^2 (-1)^{j-1} \int_S \frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_j(x')} \frac{K_{mi}(x, \xi)}{2\pi|x-\xi|^2} \Big|_{\substack{x_3=\gamma_j(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(v_x, x_3)} + \\ & + \sum_{j=1}^2 (-1)^{j-1} \int_S \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_j(x')} \frac{K_{li}(x, \xi)}{2\pi|x-\xi|^2} \Big|_{\substack{x_3=\gamma_j(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(v_x, x_3)} + \\ & - 2 \int_{\Gamma} \frac{\partial u(x)}{\partial x_i} \frac{\partial U(x-\xi)}{\partial v_x} \Big|_{\xi_3=\gamma_k(\xi')} dx - 2 \int_D \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} \frac{\partial U(x-\xi)}{\partial x_i} dx - \\ & - \int_D 2a(x)u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx \end{aligned} \quad (10)$$

and extract only singular terms ($x_3 = \xi_3$) for $k = 1,2$ then we'll get the second

necessary conditions (10) ($i = \overline{1,3}$) in the form:

$$\frac{\partial u}{\partial \xi_i} \Big|_{\xi_3=\gamma_k(\xi')} = (-1)^k \int_S \frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{K_{mi}(x, \xi)}{2\pi|x-\xi|^2} \Big|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(v_x, x_3)} +$$

$$+(-1)^{k+1} \int_S \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{K_{li}(x, \xi)}{2\pi |x-\xi|^2} \Big|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(\nu_x, x_3)} + \dots, k=1, 2, \quad (11)$$

where three dots designate the sum of nonsingular terms.

Remark 1. Three dots in (11) contain the derivatives $\frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')}$, $l=1,2,3; k=1,2$, under the sign of integral and later we shall take it into consideration.

Let us introduce the designations:

$$K_{ij}^{(k)}(x', \xi') = K_{ij}(x, \xi) \Big|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} \quad (12)$$

$$|x - \xi|^2 \Big|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} = |x' - \xi'|^2 + (\gamma_k(x') - \gamma_k(\xi'))^2 = |x' - \xi'|^2 P_k(x', \xi') \quad (13)$$

where

$$P_k(x', \xi') = 1 + \left[\frac{\partial \gamma_k(x')}{\partial x_1} \cos(x' - \xi', x_1) + \frac{\partial \gamma_k(x')}{\partial x_2} \cos(x' - \xi', x_2) + O(|x' - \xi'|) \right]^2 \neq 0, k=1, 2.$$

By means of the designations (12), (13) we'll rewrite the necessary conditions (11) for $k=1,2$ as follows:

$$\begin{aligned} & \frac{\partial u}{\partial \xi_i} \Big|_{\xi_3=\gamma_k(\xi')} = \\ & = (-1)^{(k)} \int_S \frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{1}{2\pi |x' - \xi'|^2} \frac{K_{mi}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + \\ & + (-1)^{(k+1)} \int_S \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{1}{2\pi |x' - \xi'|^2} \frac{K_{li}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + \dots, i = \overline{1,3}; k=1,2. \end{aligned} \quad (14)$$

Thus, we have established the following

Theorem 2. Under assumptions of Theorem 1 necessary conditions (14) are singular.

4. Regularization of the necessary conditions

Let us build a linear combination of necessary conditions (14) for $k=1,2$ ($i=1,2$; $j=1,2,3$) with unknown yet coefficients $\beta_{ij}^{(k)}(\xi')$ and bracket the common factor

$\frac{1}{2\pi|x'-\xi'|^2}$ under the sign of integral ($i=1,2$):

$$\begin{aligned} & \sum_{j=1}^3 \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_2(\xi')} \right) = \\ & = \int_s \frac{1}{2\pi|x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^2 (-1)^k \times \\ & \times \sum_{j=1}^3 \beta_{ij}^{(k)}(\xi') \left(\frac{\partial u(x)}{\partial x_m} \Big|_{x_3=\gamma_k(x')} \frac{K_{mj}^{(k)}(x', \xi')}{P_k(x', \xi')} + \frac{\partial u(x)}{\partial x_l} \Big|_{x_3=\gamma_k(x')} \frac{K_{lj}^{(k)}(x', \xi')}{P_k(x', \xi')} \right) + \dots \quad (15) \end{aligned}$$

Adding and subtracting $\beta_{ij}^{(k)}(x')$ from $\beta_{ij}^{(k)}(\xi')$, $k=1,2$ in (15) and suggesting that functions $\beta_{ij}^{(k)}(\xi')$ satisfy Hölder condition we get a weak singularity in the integrals with $\frac{\beta_{ij}^{(k)}(\xi') - \beta_{ij}^{(k)}(x')}{|x'-\xi'|^2}$. Expanding all the coefficients at the derivatives

by Taylor's formula at point $\xi' = x'$:

$$\frac{K_{ij}^{(k)}(x', \xi')}{P_k(x', \xi')} = \frac{K_{ij}^{(k)}(x', x')}{P_k(x', x')} + \frac{\partial}{\partial x_1} \left(\frac{K_{ij}^{(k)}(x', x')}{P_k(x', x')} \right) (x_1 - \xi_1) + \dots$$

and discarding the terms with weak singularity we obtain:

$$\begin{aligned} & \sum_{j=1}^3 \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_2(\xi')} \right) = \\ & = \int_s \frac{1}{2\pi|x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} \sum_{k=1}^2 (-1)^k \times \\ & \times \sum_{j=1}^3 \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_k(x')} \left(\beta_{il}^{(k)}(x') \frac{K_{lj}^{(k)}(x', x')}{P_k(x', x')} + \beta_{im}^{(k)}(x') \frac{K_{mj}^{(k)}(x', x')}{P_k(x', x')} \right) + \dots \quad (16) \end{aligned}$$

where $i=1,2$ and the numbers j, l, m form a permutation of numbers 1,2,3.

To regularize the integral in the right hand side of (16) let us impose conditions on the coefficients $\beta_{ij}^{(k)}(\xi')$, i.e. let the coefficients at the derivatives under the sign of integral (16) be equal to the coefficients $\alpha_{ij}^{(k)}(\xi')$ from the boundary conditions (2). Then we get a system of 6 equations for each $i=1,2$:

$$(-1)^k \beta_{il}^{(k)}(x') \frac{K_{lj}^{(k)}(x', x')}{P_k(x', x')} + (-1)^k \beta_{im}^{(k)}(x') \frac{K_{mj}^{(k)}(x', x')}{P_k(x', x')} = \alpha_{ij}^{(k)}(x'), \quad (17)$$

$$k=1,2; j=1,2,3,$$

where the numbers j, l, m form a permutation of numbers 1,2,3 as we mentioned above.

We assume that system (17) has the unique solution $\beta_{ij}^{(k)}(x'), i, k = 1,2; j = 1,2,3$.

Remark 2. As the system (17) is linear the obtained functions $\beta_{ij}^{(k)}(x'), i, k = 1,2; j = 1,2,3$, are linear with respect to the given functions $\alpha_{ij}^{(k)}(x'), i, k = 1,2; j = 1,2,3$, and, therefore, satisfy a Hölder condition.

Then for the further regularization we replace the expression under the integral sign in the right-hand side of (17) using boundary conditions (2):

$$\sum_{j=1}^3 \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_2(\xi')} \right) =$$

$$= \int_S \frac{\varphi_i(x')}{2\pi |x' - \xi|^2} \frac{dx'}{\cos(\nu_x, x_3)} - \int_S \frac{1}{2\pi |x' - \xi|^2} \left[\sum_{k=1}^2 \alpha_i^{(k)}(x') u(x', \gamma_k(x')) \right] \frac{dx'}{\cos(\nu_x, x_3)} \dots$$

$$. \quad (18)$$

From necessary condition (7) for $u(\xi)$ on $\Gamma_k, k = 1,2$, discarding the term with the normal derivative $\frac{\partial u}{\partial \nu_x}$ in the integrand and leaving only weakly singular terms we have:

$$u(\xi) \Big|_{\xi_3=\gamma_k(\xi')} = - \int_S \frac{u(x) \Big|_{x_3=\gamma_k(x')}}{2\pi|x'-\xi'|^2} \frac{\cos(x-\xi, \nu_x) \Big|_{\substack{\xi_3=\gamma_k(\xi') \\ x_3=\gamma_k(x')}}}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + \dots \quad (19)$$

Substituting necessary conditions (19) into (18) and changing the order of integration we'll obtain regularized relationships:

$$\begin{aligned} & \sum_{j=1}^3 \left(\beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \Big|_{\xi_3=\gamma_2(\xi')} \right) = \\ & = - \sum_{k=1}^2 \int_S \frac{u(\zeta) \Big|_{\zeta_3=\gamma_k(\zeta')}}{\cos(\nu_\zeta, \zeta_3)} d\zeta' \int_S \alpha_i^{(k)}(x') \frac{\cos(\zeta-\xi, \nu_\zeta) \Big|_{\substack{\zeta_3=\gamma_k(\zeta') \\ x_3=\gamma_k(x')}}}{2\pi|x'-\zeta'|^2|x'-\xi'|^2 P_k(x', \zeta')} \frac{dx'}{\cos(\nu_x, x_3)} + \\ & \quad + \int_S \frac{\varphi_i(x')}{2\pi|x'-\xi'|^2} \frac{dx'}{\cos(\nu_x, x_3)} + \dots, \quad (20) \end{aligned}$$

as the interior integral in the first term in the RHS of (20) doesn't contain the unknown and the second term is convergent if functions $\varphi_i(x')$, $i=1,2$, are continuously differentiable in S and vanish on the boundary $\partial S = \bar{S} \setminus S$.

Thus, we have established the following

Theorem 3. *Let the conditions of Theorem 1 hold true. If system (17) is uniquely resolved, the conditions (2) are linear independent, the coefficients $\alpha_{ij}^{(k)}(x')$ for $i=1,2; j=1,3; k=1,2$, belong to some Hölder class and the rest of the coefficients and kernels are continuous functions, functions $\varphi_i(x')$, $i=1,2$, are continuously differentiable and vanish on the boundary $\partial S = \bar{S} \setminus S$ then the relationships (20) are regular.*

5. Fredholm property of the problem

It is well-known from the calculus that

$$\frac{\partial}{\partial x_p} u(x_1, x_2, \gamma_k(x_1, x_2)) = \frac{\partial u(x)}{\partial x_p} \Big|_{x_3=\gamma_k(x')} + \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_k(x')} \frac{\partial \gamma_k(x')}{\partial x_p}, \quad k=1,2; \\ p=1,2,$$

whence we have

$$\frac{\partial u(x)}{\partial x_p} \Big|_{x_3=\gamma_k(x')} = \frac{\partial u(x', \gamma_k(x'))}{\partial x_p} - \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_k(x')} \frac{\partial \gamma_k(x')}{\partial x_p}, \quad p=1,2; k=1,2. \quad (21)$$

So, the derivatives $\frac{\partial u(x)}{\partial x_1} \Big|_{x_3=\gamma_k(x')}$ and $\frac{\partial u(x)}{\partial x_2} \Big|_{x_3=\gamma_k(x')}$ are defined through

the derivative $\frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_k(x')}$. Then we have two unknown quantities: the

boundary values of the unknown function $u(x', \gamma_1(x'))$ and $u(x', \gamma_2(x'))$.

After substituting (21) for $\frac{\partial u(x)}{\partial x_1} \Big|_{x_3=\gamma_k(x')}$ and $\frac{\partial u(x)}{\partial x_2} \Big|_{x_3=\gamma_k(x')}$ into (2) and

grouping the unknown quantities we get a system:

$$\begin{aligned} l_i u = & \sum_{m=1}^2 \sum_{j=1}^2 \alpha_{ij}^{(m)}(x') \left(\frac{\partial u(x', \gamma_m(x'))}{\partial x_j} - \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_m(x')} \frac{\partial \gamma_m(x')}{\partial x_j} \right) + \\ & + \sum_{m=1}^3 \alpha_{i3}^{(m)}(x') \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_m(x')} + \\ & + \sum_{m=1}^2 \alpha_i^{(m)}(x') u(x', \gamma_m(x')) = \varphi_i(x'), \quad x' \in S, \quad i = 1, 2. \end{aligned} \quad (22)$$

Introducing the designations:

$$A_{ij}(x') = \alpha_{i3}^{(j)}(x') - \sum_{m=1}^2 \alpha_{im}^{(j)}(x') \frac{\partial \gamma_j(x')}{\partial x_m}, \quad i, j = 1, 2,$$

system (22) can be rewritten in the form:

$$A_{i1}(x') \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_1(x')} + A_{i2}(x') \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_2(x')} = F_i(x'), \quad i = 1, 2, \quad (23)$$

where the right-hand sides of system (23) have the form:

$$F_i(x') = \varphi_i(x') - \sum_{j=1}^2 \sum_{m=1}^2 \alpha_{ij}^{(m)}(x') \frac{\partial u(x', \gamma_m(x'))}{\partial x_j} +$$

$$+ \sum_{k=1}^2 \alpha_i^{(k)}(x') u(x', \gamma_k(x')), \quad x' \in S, i = 1, 2. \quad (24)$$

Remark 3. Note that the RHSs $F_i(x')$ of system (23) are, in the virtue of (24),

functionals of $u|_{\Gamma_1}, u|_{\Gamma_2}$ and $\frac{\partial u|_{\Gamma_k}}{\partial x_j}, k, j = 1, 2$:

$$F_i(x') = F_i(x', u|_{\Gamma_1}, u|_{\Gamma_2}, \frac{\partial u|_{\Gamma_1}}{\partial x_1}, \frac{\partial u|_{\Gamma_1}}{\partial x_2}, \frac{\partial u|_{\Gamma_2}}{\partial x_1}, \frac{\partial u|_{\Gamma_2}}{\partial x_2}), i = 1, 2. \quad (25)$$

If to assume that

$$\Delta(x') = \begin{vmatrix} A_{11}(x') & A_{12}(x') \\ A_{21}(x') & A_{22}(x') \end{vmatrix} \neq 0 \quad (26)$$

the system (23) can be reduced to a normal form:

$$\begin{aligned} \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_1(x')} &= \frac{1}{\Delta(x')} \begin{vmatrix} F_1(x') & A_{12}(x') \\ F_2(x') & A_{22}(x') \end{vmatrix}, \\ \frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_2(x')} &= \frac{1}{\Delta(x')} \begin{vmatrix} A_{11}(x') & F_1(x') \\ A_{21}(x') & F_2(x') \end{vmatrix}. \end{aligned}$$

whence, in the virtue of (25), we have from boundary conditions (2) and (21) :

$$\frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_k(x')} = \Phi_k(u|_{\Gamma_1}, u|_{\Gamma_2}, \frac{\partial u|_{\Gamma_1}}{\partial x_1}, \frac{\partial u|_{\Gamma_1}}{\partial x_2}, \frac{\partial u|_{\Gamma_2}}{\partial x_1}, \frac{\partial u|_{\Gamma_2}}{\partial x_2}), k = 1, 2. \quad (27)$$

Now substituting (21) for $\frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_k(x')}, j, k = 1, 2$, into regular relationships

(20) and grouping the terms we'll get a system:

$$\sum_{m=1}^2 C_{im}(\xi') \frac{\partial u(\xi)}{\partial \xi_3} \Big|_{\xi_3=\gamma_m(\xi')} = B_i(\xi'), i = 1, 2. \quad (28)$$

where

$$C_{ij}(\xi') = \left[\beta_{i3}^{(j)}(\xi') - \sum_{m=1}^2 \beta_{im}^{(j)}(\xi') \frac{\partial \gamma_j(\xi')}{\partial \xi_m} \right], i, j = 1, 2,$$

and the RHSs of (28) are as follows

$$\begin{aligned}
B_i(\xi') = & - \sum_{j=1}^2 \sum_{m=1}^2 \beta_{ij}^{(m)}(\xi') \frac{\partial u(\xi', \gamma_m(\xi'))}{\partial \xi_j} - \\
& - \sum_{k=1}^2 \int_S \frac{u(\zeta') \Big|_{\zeta_3=\gamma_k(\zeta')}}{\cos(\nu_{\zeta'}, \zeta_3)} d\zeta' \int_S \alpha_i^{(k)}(x') \frac{\cos(\zeta - \xi, \nu_{\zeta'}) \Big|_{\substack{\zeta_3=\gamma_k(\zeta') \\ x_3=\gamma_k(x')}}}{2\pi |x' - \zeta'|^2 P_k(x', \zeta')} \frac{dx'}{\cos(\nu_x, x_3)} + \\
& + \int_S \frac{\varphi_i(x')}{2\pi |x' - \xi|^2} \frac{dx'}{\cos(\nu_x, x_3)} + \dots, \xi' \in S, i = 1, 2,
\end{aligned}$$

In the virtue of remark 3.1 system (28), is a system of integral Fredholm equations of the second kind with respect to $\frac{\partial u(\xi)}{\partial \xi_3} \Big|_{\xi_3=\gamma_k(\xi')}, k = 1, 2$. Consequently, the system (28) has the unique solution

$$\frac{\partial u(\xi)}{\partial \xi_3} \Big|_{\xi_3=\gamma_k(\xi')} = \Psi_k(u|_{\Gamma_1}, u|_{\Gamma_2}, \frac{\partial u|_{\Gamma_1}}{\partial \xi_1}, \frac{\partial u|_{\Gamma_2}}{\partial \xi_1}, \frac{\partial u|_{\Gamma_1}}{\partial \xi_2}, \frac{\partial u|_{\Gamma_2}}{\partial \xi_2}), \quad (29)$$

as, evidently, $B_i(\xi'), j = 1, 2$, are linear functional of $u(\xi', \gamma_k(\xi'))$, $\frac{\partial u(\xi', \gamma_k(\xi'))}{\partial \xi_j}, j = 1, 2, k = 1, 2$.

The functional $\Phi_k, \Psi_k, k = 1, 2$, from (27) and (29) are linear with respect to the unknown values $u|_{\Gamma_1}, u|_{\Gamma_2}, \frac{\partial u|_{\Gamma_1}}{\partial \xi_j}, \frac{\partial u|_{\Gamma_2}}{\partial \xi_j}, j = 1, 2$:

$$\begin{aligned}
\Phi_k = & \sum_{i=1}^2 a_i^{(k)}(\xi') u|_{\Gamma_i} + \sum_{i,j=1}^2 b_{ij}^{(k)}(\xi') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^2 \int_S c_i^{(k)}(\zeta') u|_{\Gamma_i} d\zeta + \\
& + \sum_{i,j=1}^2 \int_S d_{ij}^{(k)}(\zeta') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} d\zeta + e_k(\xi'), k = 1, 2,
\end{aligned} \quad (30)$$

$$\Psi_k = \sum_{i=1}^2 a_i^{(l)}(\xi') u|_{\Gamma_i} + \sum_{i,j=1}^2 b_{ij}^{(l)}(\xi') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^2 \int_S c_i^{(l)}(\zeta') u|_{\Gamma_i} d\zeta +$$

$$+ \sum_{i,j=1}^2 \int_S d_{ij}^{(l)}(\xi') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} d\xi + e_l(\xi'), l = 3, 4; k = 1, 2. \quad (31)$$

Excluding $\frac{\partial u(\xi)}{\partial \xi_3} \Big|_{\xi_3 = \gamma_k(\xi')}, k = 1, 2$, from system (27), (29) we'll obtain a system

of linear integro-differential Fredholm equations of the second kind with respect to $u(\xi', \gamma_k(\xi')), k = 1, 2$:

$$\begin{aligned} \sum_{i=1}^2 A_i^{(k)}(\xi') u|_{\Gamma_i} + \sum_{i,j=1}^2 B_{ij}^{(k)}(\xi') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} + \sum_{i=1}^2 \int_S C_i^{(k)}(\xi') u|_{\Gamma_i} d\xi + \\ + \sum_{i,j=1}^2 \int_S D_{ij}^{(k)}(\xi') \frac{\partial u|_{\Gamma_i}}{\partial \xi_j} d\xi + g_k(\xi') = 0, \quad k = 1, 2, \end{aligned} \quad (32)$$

where

$$\begin{aligned} A_i^{(k)}(\xi') &= a_i^{(k)}(\xi') - a_i^{(k+2)}(\xi'), \quad B_{ij}^{(k)}(\xi') = b_{ij}^{(k)}(\xi') - b_{ij}^{(k+2)}(\xi'), \\ C_i^{(k)}(\xi') &= c_i^{(k)}(\xi') - c_i^{(k+2)}(\xi'), \quad D_{ij}^{(k)}(\xi') = d_{ij}^{(k)}(\xi') - d_{ij}^{(k+2)}(\xi'), \\ g_k(\xi') &= e_k(\xi') - e_{k+2}(\xi'), \quad k = 1, 2. \end{aligned}$$

Thus, we have come to a two-dimensional system (32) of linear integro-differential equations of the first order with Dirichlet's conditions (3) on $\partial S = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$. As the boundary ∂S is one-dimensional then this Dirichlet's condition doesn't restrict the generality because its dimension is two units less than the dimension of the domain D.

Thus, we have proved the following

Theorem 4. *If the assumptions of Theorem 3 and conditions (26) hold true and system (28) is uniquely resolved then boundary-value problem (1)-(2) is reduced to a two-dimensional system of linear integro-differential equations (32) with Dirichlet's condition (3) on the boundary $\partial S = \bar{S} \setminus S$.*

When solution $u|_{\Gamma_k}, \frac{\partial u|_{\Gamma_k}}{\partial \xi_j}, k, j = 1, 2$, is obtained from system (32), then

solution $\frac{\partial u(x)}{\partial x_3} \Big|_{x_3=\gamma_k(x')}, k=1,2$, is derived from (27), or (29). Then we find

$$\frac{\partial u(x)}{\partial x_1} \Big|_{x_3=\gamma_k(x')} \text{ and } \frac{\partial u(x)}{\partial x_2} \Big|_{x_3=\gamma_k(x')} \text{ from (21).}$$

The solution to problem (1)-(3) is obtained from the 1st and 2nd basic relationships (5) and (8) as from a system of Fredholm equations for $u(\xi)$,

$$\frac{\partial u(\xi)}{\partial \xi_i}, \xi \in D, i=\overline{1,3}:$$

$$\begin{aligned} u(\xi) = & -\sum_{j=1}^3 \sum_{k=1}^2 \int_{\Gamma_k} \left(\frac{\partial u(x)}{\partial x_j} U(x-\xi) - u(x) \frac{\partial U(x-\xi)}{\partial x_j} \right) \cos(v_x, x_j) dx - \\ & - \sum_{m=1}^3 \int_D a_m(x) \frac{\partial u(x)}{\partial x_m} U(x-\xi) dx - \int_D a(x) u(x) U(x-\xi) dx, \xi \in D, \\ \frac{\partial u(\xi)}{\partial \xi_i} = & \int_{\Gamma} \frac{\partial u(x)}{\partial x_m} \left[\frac{\partial U(x-\xi)}{\partial x_m} \cos(v_x, x_i) - \frac{\partial U(x-\xi)}{\partial x_i} \cos(v_x, x_m) \right] dx + \\ & + \int_{\Gamma} \frac{\partial u(x)}{\partial x_l} \left[\frac{\partial U(x-\xi)}{\partial x_l} \cos(v_x, x_i) - \frac{\partial U(x-\xi)}{\partial x_i} \cos(v_x, x_l) \right] dx - \\ & - \int_{\Gamma} \frac{\partial u(x)}{\partial x_i} \frac{\partial U(x-\xi)}{\partial v_x} dx - \int_D \sum_{k=1}^3 a_k(x) \frac{\partial u(x)}{\partial x_k} \frac{\partial U(x-\xi)}{\partial x_i} dx - \\ & - \int_D a(x) u(x) \frac{\partial U(x-\xi)}{\partial x_i} dx, \xi \in D, i=\overline{1,3}. \end{aligned}$$

Finally, there has been established

Theorem 5. *If the assumptions of Theorem 4 hold true then the boundary value problem (1), (2), (3) has Fredholm property.*

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