A GENERAL APPROACH TO A PRIORI ESTIMATES OF SOLUTIONS TO NONLINEAR EQUATIONS AND ORDINARY DERIVATIVE INEQUALITIES

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Abstract

This article outlines a general approach to a priori estimates of solutions to nonlinear equations and ordinary derivative inequalities, based on the method of trial functions. This approach covers a fairly wide class of nonlinear problems, for which we study the problem of the absence of nontrivial solutions (see [1]). Our approach is based on a priori estimates. First, we obtain an a priori estimate for the solution of the nonlinear problem under consideration. Then we obtain the asymptotics of this a priori estimate. The proof of the absence of a solution is carried out by contradiction. The derivation of an a priori estimate is based on the trial function method. The optimal choice of the trial function leads to a minimax nonlinear problem, which generates a nonlinear capacitance. To analyze the absence problem, it is enough to obtain an exact estimate of the first term of the asymptotics of this capacity.

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1. Introduction

Let us agree on the following notation and definitions (eg see [2, p. 10; p. 187-191]). An n dimensional Euclidean space is denoted by E_n. Let G be a set in the space E_n. The set of functions that are continuous and bounded in G will be denoted by $C(G)$.

The set of functions that have all possible derivatives in G up to order k inclusive, and these derivatives are continuous and bounded in G through $C^{(k)}(G)$. Let $C_0^k(\Omega)$, where Ω is a finite domain, denote the set of functions that are $k-$ times continuously differentiable in $\overline{\Omega}$ and vanish on $\partial\Omega$ along with all their derivatives up to order $k-1$ inclusive. Let Ω-be some domain, and $k-$ an

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integer, $0 \le k \le \infty$. Let $\mathrm{M}^{(k)}(\Omega)$ denote the set of functions that are once continuously differentiable in Ω and vanish in the boundary strip (one for each function) of the domain Ω. If Ω is an infinite domain, then we additionally require that the functions from $\mathrm{M}^{(k)}(\Omega)$ vanish outside a certain ball, also different for each function. Obviously, $\mathrm{M}^{(k)}(\Omega) \subset \mathrm{M}^{(k-1)}(\Omega)$. Functions of class $\mathrm{M}^{(\infty)}(\Omega)$ are called compactly supported in Ω. Let $a \in R_+ \equiv \{a \in R | a \ge 0\}$.
A function defined almost everywhere in some domain Ω is said to be locally integrable in Ω if it is integrable on any compact set that also contains Ω.

The set of such functions is usually denoted by $L_{loc}(\Omega)$. Let $u(x)-$ be a continuous function. The closure of the set of points at which this function is nonzero is called its support and is denoted by the symbol $\mathrm{Supp}\{\mathrm{u}\}$. Obviously, the support of the derivative of any function is contained in the support of the given function. Let us include in the set of basic functions $D = D(E_n)$ all functions that are finite and infinitely differentiable in E_n. We denote the set of basic functions whose supports are contained in the domain G by $D(G)$. Thus, $D(G) \subset D(E_n) = D$. A generalized function is any linear continuous functional on the space of basic functions D. Let us denote by $D' = D'(E_n)$ the set of all generalized functions. We will write the value of the functional (generalized function) f on the main function $\varphi(x)$ as (f,φ). We will say that the generalized function f has the order of singularity, or simply the order of j, if it can be represented in the form

$$f = \sum_{|\alpha| \le j} D^\alpha g_\alpha, \quad g_\alpha \in L_{loc}(\Omega).$$

In this case we will write $s(f) \le j$.

Obviously, the order of any locally integrable function is zero. Let f be a generalized function of order j, and φ an arbitrary basic function. Then

$$(f,\varphi) = \sum_{|\alpha| \le j} \left(D^\alpha g_\alpha, \varphi\right) = \sum_{|\alpha| \le j} (-1)^{\langle \alpha \rangle} \cdot \left(g_\alpha, D^\alpha \varphi\right) =$$
$$= \sum_{|\alpha| \le j} (-1)^{\langle \alpha \rangle} \cdot \int_\Omega g_\alpha(x) D^\alpha \varphi(x) dx, \ \ \forall \varphi \in M^{(\infty)}(\Omega). \quad (1)$$

But the right-hand side of formula (1) retains its meaning for any function $\varphi \in M^{(j)}(\Omega)$ .Using this formula, we consider the functional f for the class

$M^{(j)}(\Omega)$. From the above it follows that a generalized function of finite order j can be interpreted as distributions over the space of basic functions $D_j(\Omega)=M^{(j)}(\Omega)$.. It is appropriate to denote the class of these distributions by $D'_j(\Omega)$. It is also obvious that if $f\in D'_j(\Omega)$, then the restriction of the functional f to the set $D=M^{(\infty)}(\Omega)$ is a generalized function of class D'. With the concept of generalized solutions differential equation is closely related to the concept of generalized derivatives (see [2], p. 33). The following statements are obvious:

a) $S(f_1+f_2)=\max\big(S(f_1),S(f_2)\big)$,

b) if $\varphi\in C^{\infty}(\Omega)$, то $S(\varphi f)=S(f)$,

c) $S=(D^{\alpha}f)=S(f)+|\alpha|$.

In some domain $\Omega\subset E_n$, consider the differential equation

$$Lu=\sum_{|\alpha|=0}^{m}A_{\alpha}(x)D^{\alpha}u=f(x). \tag{2}$$

It may happen that $\Omega=E_n$. We will assume that the coefficients are $A_{\alpha}\in C^{(j+|\alpha|)}(\Omega)$, where $j-$ is an integer, $0\le j\le\infty$. We will look for solutions to equation (2) belonging to class $D'_j(\Omega)$. If $u(x)$ is such a solution, then $S(u)\le j$ and $S\big(A_{\alpha}D^{\alpha}u\big)\le j+|\alpha|$.

The left side of equation (2) is a generalized function of class $D'_{j+m}(\Omega)$. We will therefore assume that the free term of the equation is $f\in D'_{j+m}(\Omega)$. If a solution $u(x)\in D'_j(\Omega)$ exists, then $u(x)$ can be considered as a generalized solution to this equation (2). This solution is determined by the identity

$$\left(\sum_{|\alpha|=0}^{m}A_{\alpha}(x)D^{\alpha}u,\varphi\right)=(f,\varphi),\ \forall\varphi\in D_{j+m}(\Omega), \tag{3}$$

The differentiable formula for a generalized function and the rule for multiplying a generalized function of the corresponding class $C^{(l)}(\Omega)$, $l\ge\infty$, make it possible to replace relation (3) with an equivalent relation

$$\left(u,\sum_{|\alpha|=0}^{m}(-1)^{|\alpha|}D^{\alpha}\big(A_{\alpha}\varphi\big)\right)=(f,\varphi),\ \forall\varphi\in D_{j+m}(\Omega).$$

Thus, a locally integrable generalized solution to equation (2) can be

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treated as a regular generalized function, which is a generalized solution of the same equation.

Consider an ordinary differential inequality $k-$ of order

$$\begin{cases} \dfrac{d^k u}{dt^k} + a_{k-1}(t)\dfrac{d^{k-1}u}{dt^{k-1}} + \ldots + a_0(t)u \geq b(t)\cdot|u|^q,\ t \geq 0, \\ u^{(k-1)}(0) = u_{k-1} > 0 \end{cases} \tag{4}$$

C $a_0,\ldots,a_{k-1},b \in L^1_{loc}(R_+)$ and $q > 1,\ b > 0$ in R_+.

Definition. By a weak solution to problem (4) we mean a function $u(x) \in W^1_{q,loc}([0,+\infty))$ satisfying the inequality

$$\int_0^{T_\bullet} b(t)|u|^q \varphi(t)dt \leq (-1)^k \cdot \int_0^{T_\bullet} u(t)\frac{d^k\varphi}{dt^k}dt - u_{k-1} + \int_0^{T_\bullet} u(t)L\varphi(t)dt$$

for any function $\varphi(t) \geq 0$ from class $D_{j+k}(R_+)$.

We multiply inequality (4) by a test function $\varphi(t) \geq 0$ from class $D_{j+k}(R_+)$ such that

$$\varphi'(0) = \varphi''(0) = \ldots = \varphi^{(k-1)}(0) = 0$$

and

$$\varphi(T_1) = \varphi'(T_1) = \varphi''(T_1) = \ldots = \varphi^{(k-1)}(T_1) = 0,$$

where

$$\varphi(t) = \begin{cases} 1, & 0 \leq t \leq T < T_1, \\ 0, & t \geq T_1. \end{cases}$$

Then we get

$$\int_0^{T_\bullet} b(t)|u|^q \varphi(t)dt \leq \int_0^{T_\bullet} \frac{d^k u}{dt^k}\varphi dt + \int_0^{T_\bullet} a_{k-1}(t)\frac{d^{k-1}u}{dt^{k-1}}\varphi dt + \ldots + \int_0^{T_\bullet} a_0(t)u(t)\varphi(t)dt =$$

$$= (-1)^k \cdot \int_0^{T_\bullet} u(t)\frac{d^k\varphi}{dt^k}dt - u_{k-1} + \int_0^{T_\bullet} u(t)L\varphi dt,$$

where $L\varphi(t) = (-1)^{k-1}\dfrac{d^{k-1}}{dt^{k-1}}\left(a_{k-1}(t)\varphi\right) + \ldots + (-1)^1\dfrac{d}{dt}\left(a_1(t)\varphi(t)\right) + a_0(t)\varphi(t).$

Hence, due to Young's inequality with parameter $\varepsilon > 0$

$$a \cdot b \leq \frac{\varepsilon}{r}a^r + \frac{1}{r' \cdot \varepsilon^{r'-1}}b^{r'},\ a,b \geq 0,$$

where $r' = \frac{r}{r-1}$. We get

$$\int_0^{T_\bullet} b(t)|u|^q \varphi(t)dt \le \int_0^{T_\bullet} u(t)\cdot\left|\varphi^{(k)}\right|dt + \int_0^{T_\bullet} |u| : |L\varphi|dt - u_{k-1} = \int_0^{T_\bullet} |u|\left[\left|\varphi^{(k)}\right| + |L\varphi|\right]dt - u_{k-1} =$$

$$= \int_0^{T_\bullet} |u| L^*\varphi dt - u_{k-1} = \int_0^{T_\bullet} |u|[b(t)\cdot\varphi(t)]^{1/q}\cdot[b(t)\cdot\varphi(t)]^{-1/q}\cdot L^*\varphi dt - u_{k-1} \le$$

$$\le \left(\int_0^{T_\bullet} |u|^q \cdot b(t)\varphi(t)dt\right)^{1/q}\cdot\left(\int_0^{T_\bullet} \frac{\left|L^*\varphi\right|^{q'}}{[b(t)\cdot\varphi(t)]^{q'/q}}dt\right)^{1/q'} - u_{k-1} \le$$

$$\le \frac{\varepsilon}{q}\int_0^{T_\bullet} b(t)|u|^q\varphi(t)dt + \frac{1}{q'\varepsilon^{q'-1}}\cdot\int_0^{T_\bullet}\frac{\left|L^*\varphi\right|^{q'}}{[b(t)\cdot\varphi(t)]^{q'-1}} - u_{k-1}.$$

$$\left(1-\frac{\varepsilon}{q}\right)\int_0^{T_\bullet} b(t)|u|^q\varphi(t)dt \le \frac{1}{q'\varepsilon^{q'-1}}\cdot\int_0^{T_\bullet}\frac{\left|L^*\varphi\right|^{q'}dt}{[b(t)\cdot\varphi(t)]^{q'-1}} - u_{k-1},$$

where $L^*\varphi = \left|\varphi^{(k)}\right| + |L\varphi|$.

Thus, for any $q > \varepsilon > 0,\ q > 1,$ we obtain an a priori estimate that does not depend on the initial values of $u(0),\dots,u^{(k-2)}(0)$.

From here we get

$$\int_0^{T_\bullet} b(t)|u|^q\varphi(t)dt \le \int_0^{T_\bullet}\frac{\left|L^*\varphi\right|^{q'}dt}{[b(t)\cdot\varphi(t)]^{q'-1}} - q'\cdot u_{k-1},$$

since $(q>1)$ $\min\limits_{o<\varepsilon<q}\left\{\frac{q-1}{q-\varepsilon}, \frac{1}{q'\varepsilon^{q'-1}}\right\} = 1$ is achieved at $\varepsilon = 1$.

To obtain an optimal a priori estimate, we introduce the following quantity

$$cap\left(D^k, T\right) = \inf_{T_\bullet > T}\left\{\int_0^{T_\bullet}\frac{\left|L^*\varphi\right|^{q'}dt}{[b(t)\cdot\varphi(t)]^{q'-1}}\right\},$$

where the infimum is taken over all test functions $\varphi(t)$ from the specified class. It is natural to call this quantity the nonlinear capacity induced by our problem. Then the optimal a priori estimate takes the form

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$$\int_0^{T} b(t)|u|^q \varphi(t)dt \le cap\left(D^k, T\right).$$

Let us take as a test function $\varphi(t)$ a function of the form

$$\varphi(t) = \varphi_0(\tau), \;\; \tau = \frac{t}{T},$$

where $\varphi_0 \in C_0^k(R), \;\; \varphi_0 \ge 0$ and such that

$$\varphi_0(\tau) = \begin{cases} 1, & 0 \le \tau \le 1, \\ 0, & \tau \le \tau_1 > 1. \end{cases}$$

Then

$$\int_0^{T} b(t)|u|^q \varphi dt \le \frac{1}{T^{kq'-1}} \int_1^{T} \frac{\left|L^* \varphi_0\right|^{q'} dt}{\left[\varphi_0(\tau) b(\tau T)\right]^{q'-1}} - q' \cdot u_{k-1}.$$

It is clear that the function $\varphi_0(\tau)$ from the class C under consideration

$$\int_1^{T} \frac{\left|L^* \varphi_0\right|^{q'} dt}{\left[\varphi_0(\tau) b_0(\tau T)\right]^{q'-1}} < \infty$$

exists.

Let us denote the value of this integral by $c_1 > 0$. Then we get

$$\int_0^{T} b(t)|u|^q \varphi dt \le c_1 T^{1-kq'} - q' \cdot u_{k-1}.$$

From this estimate for $u_k \ge 0$ we obtain the absence of a global nontrivial solution for $kq' > 1,$ i.e. for any $k \ge 1$ and $q > 1.$

If $u_{k-1} > 0,$ then at $T > T_0$ c

$$c_1 \cdot \frac{1}{T_0^{\,kq'-1}} - q' u_{k-1} = 0,$$

$$T_0^{\,kq'-1} = \frac{c_1}{q' u_{k-1}},$$

$$T_0 = \left(\frac{c_1}{q' u_{k-1}}\right)^{\frac{1}{kq'-1}}$$

there is no solution to the problem under consideration.

Remark. The dependence of the lifetime on the initial value of $u_{k-1} > 0$ is exact, i.e. unimprovable in the entire class of problems under consideration. Obtaining an exact constant $c_1 > 0$ involves finding

$$\inf_{\tau_{\bullet}>1} \int_1^{\tau_{\bullet}} \frac{\left|L^* \varphi\right|^{q'} d\tau}{\left[b(\tau T) \cdot \varphi_0(\tau)\right]^{q'-1}}$$

in the class of function φ_0 under consideration.

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