*Baku State University Journal of Mathematics & Computer Sciences **2024**, v. 1 (3), p. 1-9*

journal homepage: http://bsuj.bsu.edu.az/en

A NECESSARY CONDITION AND A SUFFICIENT CONDITION FOR THE SUMMABILITY OF THE DISCRETE RIESZ TRANSFORM

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Received 20 June 2024; accepted 10 September 2024

https://doi.org/10.30546/209501.101.2024.1.3.09

Abstract

The Riesz transform has been well studied on classical Lebesgue, Morrey, Sobolev, Besov, Campanato, etc. spaces. But its discrete version has not been well studied. In this paper, we find a necessary condition and a sufficient condition for the summability of the discrete Riesz transform.

Keywords: Riesz transform, discrete Riesz transform, Lebesgue spaces, summability, distribution function.

Mathematics Subject Classification (2020): 44A15, 46B45, 42B35.

1. Introduction

The j-th Riesz transform of the function $f \in L_p(R^d)$, $1 \le p < +\infty$ is defined as the following singular integral:

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$$R_j(f)(x) = v.p.\int_{R^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy = \gamma_{(d)} \lim_{\varepsilon\to 0} \int_{\{y\in R^d : |x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy,$$

$$j = 1, 2, \ldots, d .$$

where $\gamma_{(d)} = \dfrac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ is Euler's Gamma function.

It is well known (see [9, 13, 17, 18]) that the Riesz transform plays an important role in the theory of harmonic functions. The boundary values of harmonically conjugate in the upper half space functions are interconnected by the Riesz transform.

From the theory of singular integrals (see [17]) it is well known that the Riesz transform is a bounded operator in the space $L_p(R^d)$, $p > 1$, that is, if $f \in L_p(R^d)$, then $R_j(f) \in L_p(R^d)$ and the inequality

$$\|R_j f\|_{L_p} \le C_p \|f\|_{L_p} \tag{1}$$

holds. In the case $f \in L_1(R^d)$ only the weak inequality holds:

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} \le \frac{C_1}{\lambda}\|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f. From the inequalities (1), (2) it follows that the Riesz transform of the function $f \in L_1(R^d)$ satisfies the condition

$$m\{x \in R^d : |(R_j f)(x)| > \lambda\} = o(1/\lambda), \quad \lambda \to +\infty.$$

Note that the Riesz transform of a function $f \in L_1(R^d)$ is generally not Lebesgue integrable. In [7], using the notion A-integrability of functions, an analogue of the Riesz equation for the Riesz transform of functions from the class $L_1(R^d)$ was proved. In [10-12, 14-17], the boundedness of the Riesz transform in the functional spaces of Sobolev, Besov, Orlicz, Companato, Morrey, etc. was studied. But the discrete analogue of the Riesz transform has not been fully studied. In this paper, we find a necessary condition and a sufficient condition for the summability of the discrete Riesz transform.

2. Discret Riesz transform and its properties

Denote by $l_p := l_p\left(Z^d\right)$, $p \ge 1$, the class of sequences $h = \{h_m\}_{m \in Z^d}$, satisfying the
condition

$$\|h\|_{l_p} := \left(\sum_{m \in Z^d} |h_m|^p \right)^{1/p} < \infty ,$$

where $Z^d := \left\{ m = \left(m_1, \ldots, m_k\right): \; m_i \in Z, \, i = \overline{1, d} \right\}$ and Z is the set of integers. Let $h = \{h_m\}_{m \in Z^d} \in l_p$, $p \ge 1$. Namely, the sequence $\tilde{R}_j(h) = \left\{\left(\tilde{R}_j h\right)_n\right\}_{n \in Z^d}$ is called the Riesz transform of the sequence h, where

$$\left(\tilde{R}_j h\right)_n = \sum_{m \in Z^d, \; m \ne n} \frac{n_j - m_j}{|n - m|^{d+1}} \cdot h_m \, , \quad n = \{n_1, \ldots, n_d\} \in Z^d .$$

Note that if $h \in l_p$, $1 \le p < \infty$, then it follows from the Holder inequality that the series $\sum_{m \in Z^d, \; m \ne n} \frac{n_j - m_j}{|n - m|^{d+1}} \cdot h_m$ absolutely converges, and therefore the Riesz transform of the sequence h exists.

Let's note some properties of the discrete Riesz transform, obtained in the work [1].

Theorem 1 [1]. Let $1 < p < \infty$. For any $h \in l_p$ we have $\tilde{R}_j h \in l_p$, and there exist $c_p > 0$ such that

$$\left\|\tilde{R}_j h\right\|_{l_p} \le c_p \cdot \|h\|_{l_p} .$$

Theorem 2 [1]. There exist $c_1 > 0$ such that for any $h \in l_1$ and for any $\lambda > 0$ the distribution function

$$\left(\widetilde{R}_j h\right)(\lambda) = \left|\left\{n \in Z^d : \left|\left(\widetilde{R}_j h\right)_n\right| > \lambda\right\}\right| := \sum_{\left\{n \in Z^d : \left|\left(\widetilde{R}_j h\right)_n\right| > \lambda\right\}} 1$$

of the Riesz transform of the sequence h satisfies the inequality

$$\left|\left(\widetilde{R}_j h\right)(\lambda)\right| \le \frac{c_1}{\lambda} \|h\|_{l_1}.$$

Theorem 3 [1]. Let $h \in l_1$. Then the equation

$$\lim_{\lambda \to 0+} \lambda \cdot \left(\widetilde{R}_j h\right)(\lambda) = \mu_{(d)} \left|\sum_{n \in Z^d} h_n\right|$$

holds, where

$$\mu_{(d)} = \frac{2^d}{d \cdot (d-1)!!} \left(\frac{\pi}{2}\right)^{\left[\frac{d-1}{2}\right]}$$

and $\left[\frac{d-1}{2}\right]$ is integer part of a number $\frac{d-1}{2}$.

In addition, we note that the boundedness of the discrete Hilbert, discrete Ahlfors-Beurling and discrete Riesz transforms on discrete Morrey spaces was studied in [2, 4, 5, 8].

3. A necessary condition and a sufficient condition for the summability of the discrete Riesz transform

Theorem 4. Let $h \in l_1$. Then to include $\widetilde{R}_j h \in l_1$, it is necessary that the equation

$$\sum_{n \in Z^d} h_n = 0 \tag{3}$$

holds.

Proof. In first we prove that if the sequence $b = \{b_n\}_{n \in Z^d} \in l_1$, then the distribution function $b(\lambda)=\left|\left\{n\in Z^d : |b_n|>\lambda\right\}\right|$ of the sequence b satisfies the condition

$$b(\lambda)=o(1/\lambda),\ \ \lambda\to 0+. \tag{4}$$

It follows from the inequality

$$\sum_{n\in Z^d}|b_n|=\sum_{\{n\in Z^d:|b_n|>1\}}|b_n|+\sum_{k=0}^{\infty}\left[\sum_{\{n\in Z^d:|b_n|\in(2^{-k-1};2^{-k}]\}}|b_n|\right]\geq$$

$$\geq\left|\left\{n\in Z^d:|b_n|>1\right\}\right|+\sum_{k=0}^{\infty}\left[2^{-k-1}\cdot\left|\left\{n\in Z^d:|b_n|\in\left(2^{-k-1};2^{-k}\right]\right\}\right|\right]=$$

$$=b(1)+\sum_{k=0}^{\infty}\left[2^{-k-1}\cdot\left(b\left(2^{-k-1}\right)-b\left(2^{-k}\right)\right)\right]=\sum_{k=0}^{\infty}\left[2^{-k-1}\cdot b\left(2^{-k}\right)\right]$$

that

$$\lim_{k\to\infty}2^{-k}\cdot b\left(2^{-k}\right)=0.$$

Hence, taking into the decreasing of the function $b(\lambda)$, we obtain (4).

It follows from (4) that, if $\widetilde{R}_j h\in l_1$, then

$$\left(\widetilde{R}_j h\right)(\lambda)=o(1/\lambda),\ \ \lambda\to 0+,$$

and, therefore, by Theorem 3, we obtain that the equation (4) holds. The proof of the theorem 4 is complete.

Theorem 5. If the sequence $h\in l_1$ satisfies the conditions

i) $\sum_{n\in Z^d}h_n=0$;

ii) $\sum_{m\in Z^d}|h_m|\ln(e+|m|)<\infty$,

then $\widetilde{R}_j h\in l_1$ and the inequality

$$\left\|\widetilde{R}_j h\right\|_{l_1}\leq(d+5)2^{3d+1}\sum_{m\in Z^d}|h_m|\ln(e+|m|) \tag{5}$$

holds.

Proof. From the definition of the discrete Riesz transform it follows that

$$\left|\left(\tilde{R}_j h\right)_0\right| = \left|\sum_{m\neq 0} \frac{h_m}{|m|^d}\right| \leq \|h\|_{l_1}. \tag{6}$$

From the condition i) for $n \neq 0$ we have

$$\left|\left(\tilde{R}_j h\right)_n\right| = \left|\sum_{m\in Z^d,\ m\neq n} \frac{n_j - m_j}{|n-m|^{d+1}} \cdot h_m\right| = \left|\sum_{m\in Z^d,\ m\neq n} \frac{n_j - m_j}{|n-m|^{d+1}} \cdot h_m - \sum_{m\in Z^d} \frac{n_j}{|n|^{d+1}} \cdot h_m\right| \leq$$

$$\leq \left|\frac{h_n}{n^d}\right| + \sum_{m\in Z^d,\ m\neq n} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| \cdot h_m . \tag{7}$$

It follows from inequalities (6) and (7) that

$$\left\|\tilde{R}_j h\right\|_{l_1} = \sum_{n\in Z^d} \left|\left(\tilde{R}_j h\right)_n\right| \leq 2\|h\|_{l_1} + \sum_{n\in Z^d, n\neq 0} \left[\sum_{m\in Z^d, m\neq n} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| \cdot h_m\right] =$$

$$= 2\|h\|_{l_1} + \sum_{m\in Z^d\setminus\{0\}} |h_m| \cdot \sum_{n\in Z^d\setminus\{0\}, n\neq m} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| =$$

$$= 2\|h\|_{l_1} + \sum_{m\in Z^d\setminus\{0\}} |h_m| \cdot J_m , \tag{8}$$

where

$$J_m = \sum_{|n|\leq 3|m|, n\neq 0, n\neq m} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| + \sum_{|n|>3|m|} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| =$$

$$= J_m^{(1)} + J_m^{(2)}, \ \ m \neq 0. \tag{9}$$

Estimate the summands $J_m^{(i)}$, $m \neq 0$, $i = 1,2$. Define $k = [\log_2(4|m|)] + 1$, where $[\log_2(4|m|)]$ is the integer part of the number $\log_2(4|m|)$, we have

$$J_m^{(1)} = \sum_{|n|\leq 3|m|, n\neq 0, n\neq m} \left|\frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}}\right| \leq$$

$$\le \sum_{|n|\le 3|m|, n\ne m} \frac{1}{|n-m|^d} + \sum_{|n|\le 3|m|, n\ne 0} \frac{1}{|n|^d} \le 2 \sum_{|n|\le 4|m|, n\ne 0} \frac{1}{|n|^d} \le$$

$$\le 2\sum_{p=1}^{k} \sum_{2^{p-1}\le |n|<2^p} \frac{1}{|n|^d} \le 2\sum_{p=1}^{k} \sum_{|n|<2^p} \frac{1}{2^{d(p-1)}} \le 2\sum_{p=1}^{k} 2^{(p+1)d} \cdot \frac{1}{2^{d(p-1)}} =$$

$$= 2^{2d+1}k = 2^{2d+1}\left(\left[\log_2 |m|\right]+3\right) \le 2^{2d+3} \ln\left(e+|m|\right),$$

$$J_m^{(2)} = \sum_{|n|>3|m|} \left| \frac{n_j - m_j}{|n-m|^{d+1}} - \frac{n_j}{|n|^{d+1}} \right| =$$

$$= \sum_{|n|>3|m|} \left| n_j \left(\frac{1}{|n-m|^{d+1}} - \frac{1}{|n|^{d+1}} \right) - \frac{m_j}{|n-m|^{d+1}} \right| \le$$

$$\le \sum_{|n|>3|m|} |n_j| \cdot \frac{\left| |n|^{d+1} - |n-m|^{d+1} \right|}{|n|^{d+1}|n-m|^{d+1}} + \sum_{|n|>3|m|} \left| \frac{m_j}{|n-m|^{d+1}} \right| \le$$

$$\le \sum_{|n|>3|m|} |n| \cdot \frac{\left| |n| - |n-m| \right| \cdot \left(|n|^d + |n|^{d-1}|n-m| + \ldots + |n-m|^d \right)}{|n|^{d+1}|n-m|^{d+1}} + \sum_{|n|>3|m|} \frac{|m|}{|n-m|^{d+1}} \le$$

$$\le \sum_{|n|>3|m|} |n| \cdot \frac{|m| \cdot (d+1) \cdot \left(\frac{4}{3}|n| \right)^d}{|n|^{d+1} \left(\frac{2}{3}|n| \right)^{d+1}} + \sum_{|n|>3|m|} \frac{|m|}{\left(\frac{2}{3}|n| \right)^{d+1}} \le$$

$$\le |m| \cdot (d+2) 2^{d+1} \sum_{|n|>3|m|} \frac{1}{|n|^{d+1}} = |m| \cdot (d+2) 2^{d+1} \sum_{p=1}^{\infty} \left(\sum_{3|m|2^{p-1}<|n|\le 3|m|2^p} \frac{1}{|n|^{d+1}} \right) \le$$

$$\le |m| \cdot (d+2) 2^{d+1} \sum_{p=1}^{\infty} \frac{2^d 3^d |m|^d 2^{pd}}{3^{d+1}|m|^{d+1} 2^{(d+1)(p-1)}} \le (d+2) 2^{3d+1} \le (d+2) 2^{3d+1} \ln\left(e+|m|\right)$$

From this and from (8), (9) we obtain (5). The proof of the theorem 5 is complete.

Note that for the discrete Hilbert and the discrete Ahlfors - Beurling transform, analogues of Theorem 4 and Theorem 5 are proved in [3, 6].

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