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ON SOME ITERATIVE PROCESSES WITH RETURNED SEQUENSCES

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Abstract

In this paper we study the behaviour of the sequence of scalar (real or complex) numbers satisfying the relation $a_{n+k} = q_1 a_n + q_2 a_{n+1} + \ldots + q_k a_{n+k-1}$, where (q_k) is a fixed sequence of scalar numbers. Such kind of sequences arise in problems of analysis, fixed point theory, dynamical sistems, theory of chaos and ets. For example, investigating the spectra of triple and more than triple band triangle operator-matrices arise above mentioned sequences which required to study the behaviour of these sequences. From the point of application, the proved results and formulas in the literature for the spectra of the operator-matrices look like very complicated. In this work for the eliminating of indicated flaws we apply new approach, where the formulas for the spectra for the generalized difference operator-matrices describe circular domains, and also we study problems describing the theory of natural processes.

***Keywords**:* spectra,operator-matrix, sequence, returned sequence, population, population size.

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1. Introduction

In this subsection we summarize the knowledge of important in the existing literature concerning the spectra difference operator – matrices and their

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generalizations. The spectrum and fine spectrum of the difference operator-matrix Δ over the sequences spaces c_0 and c has been studied by B.Altay and F. Basar [1]. A.M. Akhmedov and F. Basar [2] have studied the fine spectrum of the difference operator-matrix Δ over the sequences spaces l_p and $b\upsilon_p$, where $1 \le p < \infty$. The fine spectrum of the Zweirer operator-matrix Z^s over the spaces l_1 and $b\upsilon$ has been investigated by B. Altay and M. Karakus [3]. Now we give results concerning to the spectra of some generalizations of difference operator-matrices. The fine spectrum of the generalized double-band operator $B(r,s)$ over the sequence spaces c_0 and c has been studied by B. Altay and F. Bashar [4]. Also, the fine spectrum of the operator $B(r,s)$ over the sequence spaces l_1 and $b\upsilon$ has been examined by H. Furkan and et al. [5]. The fine spectrum of the generalized difference operator Δ_v over the sequences spaces c_0 and l_1 was investigated by P.D. Srivastava and S. Kumar ([6],[7]). The fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequences spaces, c_0 , c and l_1 has been studied by A.M. Akhmedov and S.El. Shabrawy ([8],[9]). The fine spectrum of the triple-band matrices $B(r,s,t)$ over c_0 and c has been examined by H. Furkan and et al [10] over the sequence spaces l_p and $b\upsilon_p$. The spectrum of the triangular operator-matrix $D(r,0,s,0,t)$ has been examined by B.C. Tripathy and A. Paul [11], the spectrum and fine spectrum of generalized second order forward difference operator $\Delta^2_{u,\upsilon,w}$ on the sequence spaces l_1 have been studied by B.L. Panigrahi and P.D. Srivastava [12], and etc. Note that the formulas of the spectrum for double-band matrices usually describe circular domains. But for triple and more triple-band matrices the receiving formulas of spectra from the point of application looks like very complicated. We will return to this issue.

2. On the returned sequences

In this section we study special class of iterative processes arising in analysis, in fixed point theory, theory of dynamical systems , theory chaos , theory of graphs (see f.e. [13]- [19])

Definition 2.1 Let $q_1, q_2, \ldots, q_k \, (q_k \neq 0)$ be scalar numbers which satisfy the next relation

$$a_{n+k} = q_1 a_n + q_2 a_{n+1} + \ldots + q_k a_{n+k-1}, \tag{1}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of scalar numbers. We will call such sequence returned sequence of the order k.

Our aim is to study the behaviour of the sequence $\{a_n\}_{n=1}^{\infty}$. Suppose $q_1, q_2, \ldots, q_k$ are nonnegative numbers and $a_n > 0, n = 1,2,3,\ldots$.

Let us consider the next limit

$$\lim_{n\to\infty} a_{n+k} = \lim_{n\to\infty}(q_1 a_n + q_2 a_{n+1} + \ldots + q_k a_{n+k-1}). \qquad (2)$$

If the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ exist, in other words $\lim_{n\to\infty} a_n = a$, then from (2) we get

$$a = a(q_1 + q_2 + \ldots + q_k).$$

In this case, $a = 0$ or $q_1 + q_2 + \ldots + q_k = 1$.

For the simplisity we will study the behaviour of the squence in the case $k = 2$.

Consider

$$a_{n+2} = q_2 a_{n+2} + q_1 a_n \quad n \geq 1, \qquad (3)$$

$a_n > 0, n = 1,2,3,\ldots$. Let $q_1 > 0, q_2 > 0$ and $q_1 + q_2 = 1$.

If $a_1 = a_2 = b$, then

$$a_3 = q_2 a_2 + q_1 a_1 = (q_1 + q_2)b = b,$$
$$a_4 = q_2 a_3 + q_1 a_2 = (q_1 + q_2)b = b,$$

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and ets. Given sequence will be constant sequence . That's is why we will take $a_2 \neq a_1$. Without loss of generality, consider the case $a_2 > a_1$. Then we can show that,

$$a_3 = q_2 a_2 + q_1 a_1 < q_2 a_2 + q_1 a_2 = (q_1 + q_2)a_2 = a_2,$$
$$a_3 < a_2.$$

In the same time

$$a_3 = q_2 a_2 + q_1 a_1 > q_2 a_1 + q_1 a_1 = (q_1 + q_2)a_1 = a_1,$$
$$a_3 > a_1 \text{ and } a_1 < a_3 < a_2.$$

Similary we can prove that,

$$a_{2n-3} < a_{2n-1} < a_{2n-2} \;;\; a_{2n-1} < a_{2n} < a_{2n-2} \quad (n = 2,3,\ldots).$$

The last relation show that

$$[a_1, a_2] \supset [a_3, a_4] \supset \ldots \supset [a_{2n-3}, a_{2n-2}] \supset [a_{2n-1}, a_{2n}] \supset \ldots \qquad (4)$$

Also

$$a_{2n} - a_{2n-1} = q_1^{2n-2}(a_2 - a_1). \qquad (5)$$

Using the condition $0 < q_1 < 1$ we see that the length of the closed intervals (4) tends to zero. From the (5) it follows that the subsequences $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ converge to the same limit. So the sequence $\{a_n\}_{n=1}^{\infty}$ converge to that limit. It is shown that this limit is $\frac{a_2 + q_1 a_1}{1 + q_1}$ In the case $q_1 + q_2 > 1$ we may show that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges but it converges if $q_1 + q_2 < 1$ (see [16], [18]).

So we have prooved the next theorem.

Theorem 2.1 Suppose that $\{a_n\}$ is a returned sequence of the order 2 , $a_n > 0$ $n \in N$, $q_1 > 0$ and $q_2 > 0$. Then the next assertions are true:

1) if $q_1 + q_2 < 1,$ then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero;

2) if $q_1 + q_2 = 1,$ then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to $\frac{a_2 + q_1 a_1}{1 + q_1}.$

3) if $q_1 + q_2 > 1,$ then the sequence $\{a_n\}_{n=1}^{\infty}$ diverges.

The receiving results are the generalizations of known results from ([1], [2].) Note that some of above indicated assertions have generalizations for the complex case ([16], [18]). Taking into account above eliminations we have described software module corresponding to the algorithm Python.

As we expected at the performance of this program we discovered that the behavior of the sequence $\{a_n\}_{n=1}^{\infty}$ depends of the selecting the numbers q_1 and q_2. We have found that if $|q_1| + |q_2| > 1$, then the sequence $\{a_n\}_{n=1}^{\infty}$ diverges, if $|q_1| + |q_2| < 1$, then it converges (to zero). But if $|q_1| + |q_2| = 1$ then the sequence $\{a_n\}_{n=1}^{\infty}$ converges (not to zero).

3. On the problem of population processes

At the same time, when the changes occurring in the population are determined according to the recurrent relation $x_{n+1} = rx_n(1 - x_n)$ the bifurcation picture obtained as a part of the Mandelbrot set.

Here r is the growth factor. The quantity $(1 - x_n)$ is the limiting effect of the environment. x_n is a quantity measured by the ratio of the population number to the maximum possible number of the population for a given year. x_{n+1} is a quantity measured by the ratio of the population number for the next year to the maximum possible number of the population.

The expression $x_{n+1} = rx_n(1 - x_n)$ (see [19]) is called logistic regression. An arbitrary population is doomed when $r < 1$. The number in each next generation will be less than the previous generation. This will eventually lead to the destruction of the population. In the case of $r \geq 1$ the population reaches a certain stationary state. An interesting fact is that the number of possible stationary states reaches 2, when $r > 3$ (figure 3.1).

As the years pass, the number of the population in a certain year is in an upper stationary state, and in the next year it is in a lower stationary state. As we increase the quantity r, the stability values will increase and the redistribution will begin.

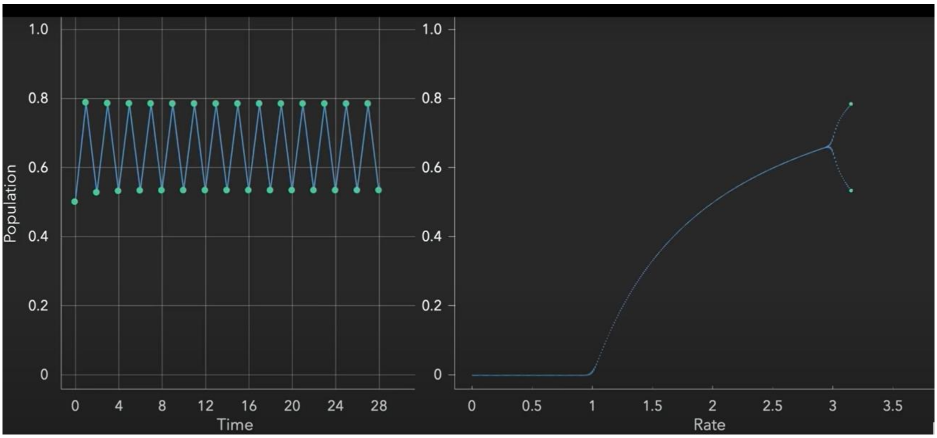

Figure 3.1

Now the cycle will consist of 4 stationary states instead of 2. The length of the cycle, in other words, the period is doubled. This is called bifurcation of peridon doubling. (figure 3.2)

As the process continues, bifurcations of this type occur again, and 8, 16, 32, 64, etc. cycles are created. Chaos occurs when $r = 3.57$.

Population does not have a stationary value. It is no coincidence that the equation

$x_{n+1} = rx_n(1-x_n)$ is the basis of pseudo-random number generator in machine technology. No apparent regularity or repetition is observed (figure 3.3;3.4), [19].

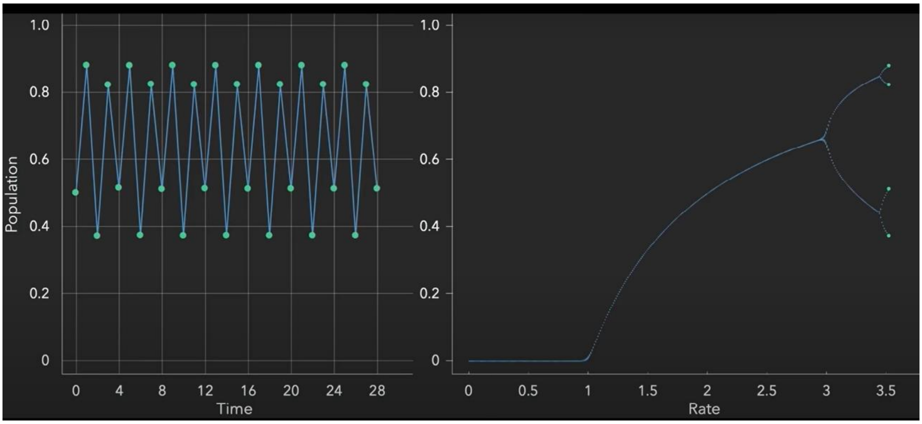

Figure 3.2

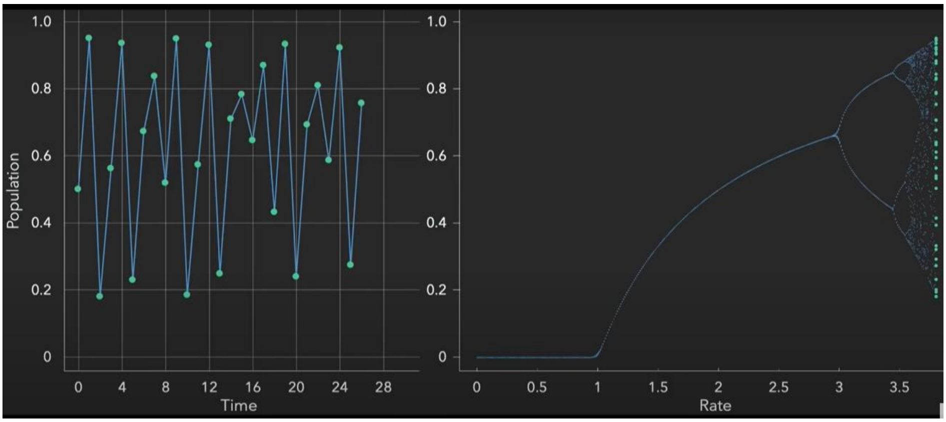

Figure 3.3

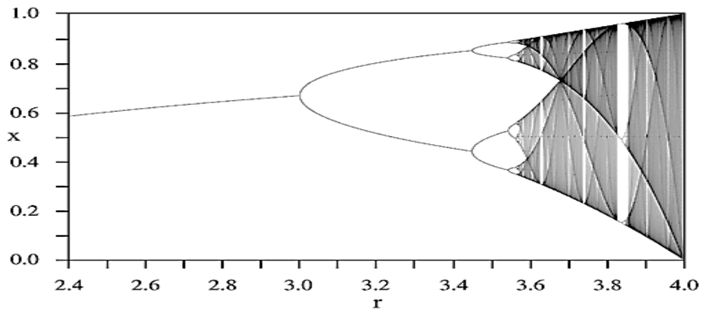

Figure 3.4

4. Fine spectrum of a class of generalized difference operator-matrices over the space c

In this subsection our aim to review some recent results concerning the spectrum of the more than that double band (triple, quadruple, and etc.) generalized difference operator-matrices acting in some sequence spaces. In such works (see e.g. [5],[10],[11],[12]) have been used the method where the main role plays the analyzing of the roots of characteristic equations of returned sequences of the order k $(k \geq 2)$, about it we've talked in introduction.

Now we give some results concerning to above indicated cases.

Denote by c_0, c, l_∞ and $b\upsilon$ (or $b\upsilon_1$) the null, convergent, bounded and bounded variation sequences spaces, respectively. Also by $l_p,\ 1 \leq p \leq \infty$ and $b\upsilon_p,\ 1 \leq p < \infty$, we denote the spaces of all p-absolutely summable and p-bounded variation sequences spaces, respectively. Main fokus in the works (see [5],[10],[11],[12]) was the triple-band matrix $B(r,s,t)$, where

$$B(r,s,t) = \begin{bmatrix} r & 0 & 0 & 0 & \ldots \\ s & r & 0 & 0 & \ldots \\ t & s & r & 0 & \ldots \\ 0 & t & s & r & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix},$$

s and t are complex parameters which do not simultaneously vanish.

The next results were received.

Theorem 4.1. $B(r,s,t) \in B(c_0)$ $(B(c))$ (see [5]) and

$$\|B(r,s,t)\|_{(c_0,c_0)} = \|B(r,s,t)\|_{(c,c)} = |r| + |s| + |t|.$$

Theorem 4.2. $B(r,s,t) \in B(l_p)$ $(B(b\upsilon_p))$ (see [10] theorem 2.1, theorem 3.1) and

$$(|r|^p + |s|^p + |t|^p)^{1/p} \le \|B(r,s,t)\|_{l_p} \le |r| + |s| + |t|.$$

$$\|B(r,s,t)\|_{bv_p} \le |r| + |s| + |t|.$$

Choosing the square roots of s the next theorem have been proved.

Theorem 4.3. ([10], theorem 2.2 theorem 3.2). Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set S by $S = \left\{ \lambda \in C : \left| \dfrac{2(r-\lambda)}{-s+\sqrt{s^2 - 4t(r-\lambda)}} \right| \le 1 \right\}$.

Then $\sigma(B(r,s,t), X) = S$, where X is one of the sequences c_0, c, l_p, $1 \le p \le \infty$ and bv_p, $1 \le p < \infty$, respectively.

The same results were received in [11, Lemmas 3;4, Theorem 5] for the artificial generalization of difference operator-matrix in c_0 and c.

As we see the formular of the spectrum very complicated. Now as an example for the operator $B(r,s,t)$ we show that it's spectrum describe circular domains in complex plane C. Suppose $|q_1| + |q_2| < 1$, where $q_1 = -\dfrac{s}{r-\lambda}$ and $q_2 = -\dfrac{t}{r-\lambda}$ then using the theorem we get

Theorem 4.4 $\sigma(B(r,s,t),c) = \{\lambda \in C |\lambda - r| \le |s| + |t|\}$ **.**

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