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ON BASICITY OF EIGENFUNCTIONS OF A SPECTRAL PROBLEM IN $L_p \oplus C$ AND L_p SPACES

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Abstract

In this paper we study the spectral problem for a discontinuous second order differential operator with a spectral parameter that arises in solving the problem of vibration of a loaded string with free ends. Using abstract theorems on the stability of basis properties of multiple systems in a Banach space with respect to certain transformations, as well as theorems on basicity of perturbed systems are proved theorems on the basicity of eigenfunctions of a discontinuous differential operator in $L_p \oplus C$ and L_p spaces.

Keywords: spectral problem; eigenfunctions; basicity.
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1. Introduction

Consider the following spectral problem with a point of discontinuity:

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$$y''(x)+\lambda y(x)=0,\ \ x\in\left(0;\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right), \tag{1}$$

$$\left.\begin{aligned}
&y'(0)=y'(1)=0,\\
&y\left(\frac{1}{3}-0\right)=y\left(\frac{1}{3}+0\right),\\
&y'\left(\frac{1}{3}-0\right)-y'\left(\frac{1}{3}+0\right)=\lambda m y\left(\frac{1}{3}\right).
\end{aligned}\right\} \tag{2}$$

Where λ is the spectral parameter, m is non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with free ends is solved by applying the Fourier method. The practical significance of such problems is noted in the well-known monographs (for example [1-3]). In [6] has asymptotical formulas for eigenvalues and eigenfunctions, and proved completeness and minimality of eigenfunctions of the problem (1), (2) in $L_p \oplus C$ and L_p spaces. The spectral problem formed by the vibration of a loaded string with fixed ends was studied in works [4, 5], [7-14]. İn these works, the asymptotics of eigenvalues and eigenfunctions were found, theorems about the completeness, minimality and basicity of eigenfunctions in various functional spaces were proved.

The spectral problems with discontinuity point and with spectral parameter in boundary conditions play an important role in mathematics, mechanics, physics and other fields of natural science, and their applications associated with the discontinuity of the physical properties of material. The study of basis properties of the spectral problems with a point of discontinuity sometimes requires completely new research methods, different from the known ones. In [15-17] new method for exploring basis properties of discontinuous differential operators has been suggested. The present paper is a continuation of the work of [6], and here using the methods of [15-17] we study the basicity of eigenfunctions and associated functions of the (1), (2) in $L_p \oplus C$ and L_p spaces.

2. Necessary information and preliminary results

For obtaining the main results we need some notions and facts from the theory of basis in a Banach space.

Definition 2.1. Let X - be a Banach space. If there exists a sequence of indexes, such that $\{n_k\} \subset N$, $n_k < n_{k+1}, n_0 = 0$, and any vector $x \in X$ is uniquely represented in the form

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i ,$$

then the system $\{u_n\}_{n \in N} \in X$ is called a basis with parentheses in X.

For $n_k = k$ the system $\{u_n\}_{n \in N}$ forms a usual basis for X.

We need the following easily proved statements.

Statement 2.1. *Let the system* $\{u_n\}_{n \in N}$ *forms a basis with parentheses for a* Banach space X. If the sequence $\{n_{k+1} - n_k\}_{k \in N}$ is bounded and the condition

$$\sup_n \|u_n\| \|\vartheta_n\| < \infty$$

holds, where $\{\vartheta_n\}_{n \in N}$ - is a biorthogonal system, then the system $\{u_n\}_{n \in N}$ forms a usual basis for X.

Statement 2.2. Let the system $\{x_n\}_{n \in N}$ forms a Riesz basis with parentheses for a Hilbert space X . If the system $\{x_n\}_{n \in N}$ uniformly minimal and almost normalized, the sequence $\{n_{k+1} - n_k\}_{k \in N}$ is bounded, then this system forms a usual Riesz basis for X.

Take the following

Definition 2.2. The basis $\{u_n\}_{n \in N}$ of Banach space X is called a p-basis, if for any $x \in X$ the condition

$$\left(\sum_{n=1}^{\infty} \left| \langle x, \vartheta_n \rangle \right|^p \right)^{\frac{1}{p}} \le M \|x\|,$$

holds, where $\{\vartheta_n\}_{n \in N}$ - is a biorthogonal system to $\{u_n\}_{n \in N}$.

Definition 2.3. The sequences $\{u_n\}_{n \in N}$ and $\{\phi_n\}_{n \in N}$ of Banach space X are called a p- close, if the following condition holds:

$$\sum_{n=1}^{\infty} \|u_n - \phi_n\|^p < \infty.$$

We will also use the following results from [17] (see also [18]).

Theorem 2.1. Let $\{x_n\}_{n\in N}$ forms a q-basis for a Banach space X, and the system $\{y_n\}_{n\in N}$ is p-close to $\{x_n\}_{n\in N}$, where $\frac{1}{p}+\frac{1}{q}=1$. Then the following properties are equivalent:

a) $\{y_n\}_{n\in N}$ - is complete in X;

b) $\{y_n\}_{n\in N}$ - is minimal in X;

c) $\{y_n\}_{n\in N}$ - forms an isomorphic basis to $\{x_n\}_{n\in N}$ for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system and $\{\hat{\vartheta}_n\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1},...,\alpha_{nm}); \ \hat{\vartheta}_n = (\vartheta_n; \beta_{n1},...,\beta_{nm}).$$

Let $J = \{n_1,...,n_m\}$ some set of m natural numbers. Suppose

$$\delta = \det\|\beta_{n_i j}\|_{i,j=\overline{1,m}}.$$

In [19] (see also [20]) has been proved the following theorem :

Theorem 2.2. Let the system $\{\hat{u}_n\}_{n\in N}$ forms a basis for X_1. In order to the system $\{u_n\}_{n\in N_j}$ where $N_J = N \setminus J$ forms a basis for X it is necessary and sufficient that the condition $\delta \neq 0$ must be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & \cdots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \cdots & \beta_{n_m 1} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{nm} & \beta_{n_1 m} & \cdots & \beta_{n_m m} \end{vmatrix}.$$

In particular, if X is a Hilbert space and the system $\{u_n\}_{n\in N}$ forms a Riesz basis for X_1, then under the condition , the system $\{u_n\}_{n\in N_j}$ also forms a Riesz basis for X . For $\delta = 0$ the system $\{u_n\}_{n\in N_j}$ is not complete and is not minimal in X.

Let X be a Banach space and the system $\{u_{kn}\}_{k=\overline{1,m},;n\in N}$ is any system in X. Let $a_{ik}^{(n)},\ i,k=\overline{1,m},\ n\in N$ any complex numbers. Let

$$A_n = \left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}} \text{ and } \Delta_n = \det A_n,\ n\in N.$$

Consider the following system in space X :

$$\hat{u}_{kn} = \sum_{i=1}^{m} a_{ik}^{(n)} u_{in},\ k=\overline{1,m}; n\in N. \quad (3)$$

Following theorems have been proved in [13] (also [15,16])

Theorem 2.3. If the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis for X and

$$\Delta_n \neq 0,\ \forall n\in N,\ (4)$$

then the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms basis with parentheses for X. If in additon the following conditions

$$\sup_n \left\{\|A_n\|, \left\|A_n^{-1}\right\|\right\} < \infty,\ \sup_n \left\{\|u_{kn}\|, \|\vartheta_{kn}\|\right\} < \infty \ (5)$$

hold, where $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N} \subset X^* -$ is biorthogonal to $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual basis for X.

Theorem 2.4. If X -is a Hilbert space, and the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms a Riesz basis for X, for holding the condition (4) the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a Riesz basis with parentheses for X. If in addition the condition (5) holds, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual Riesz basis for X.

3. Main results

İn [6] it was proved that the eigenvalues of the problem (1),(2) are asymptotically simple and consist of $\lambda_0 = 0$ and two series: $\lambda_{i,n} = \rho_{i,n}^2,\ i=1,2;$ $n\in Z^+$ where $Z^+ = \{0\}\cup N$ and the numbers $\rho_{i,n}$ hold the following asymptotically formulas:

$$\begin{cases} \rho_{1,n} = 3\pi n + \dfrac{3\pi}{2} + O\left(\dfrac{1}{n}\right) \\ \rho_{2,n} = \dfrac{3\pi n}{2} + \dfrac{3\pi}{4} + O\left(\dfrac{1}{n}\right). \end{cases} \tag{6}$$

Also the eigenfunctions $y_0(x)$ and $y_{i,n}(x)$ of the problem (1),(2) corresponding to the eigenvalues $\lambda_{i,n} = \rho_{i,n}^2,\ i = 1,2;\ n \in Z^+$ hold the following formulas

$$y_0(x) \equiv 0,\ y_{i,n}(x) = \begin{cases} \cos\dfrac{2\rho_{i,n}}{3}\cos\rho_{i,n}x, & x \in \left[0, \dfrac{1}{3}\right], \\ \cos\dfrac{\rho_{i,n}}{3}\cos\rho_{i,n}(1-x), & x \in \left[\dfrac{1}{3}, 1\right], \end{cases} \quad i = 1,2;\ n \in Z^+ \tag{7}$$

Now consider a problem on basicity of eigen- and associated functions of the problem (1),(2) in spaces $L_p(0,1) \oplus C$ and $L_p(0,1)$. Since the eigenvalues are asymptotically simple, the problem can only have a finite number of associated functions. In [6] we were constracted linearizing operator as following form. By $W_p^k\left(0, \frac{1}{3}\right) \oplus W_p^k\left(\frac{1}{3}, 1\right)$ we denoted a space functions whose contractions on segments $\left[0, \frac{1}{3}\right]$ and $\left[\frac{1}{3}, 1\right]$ belong correspondingly to Sobolev spaces $W_p^k\left(0, \frac{1}{3}\right)$ and $W_p^k\left(\frac{1}{3}, 1\right)$. Let us define the operator L in $L_p(0,1) \oplus C$ as follows:

$$D(L) = \begin{cases} \hat{y} = L_p(0,1) \oplus C : \hat{y} = \left(y, my\left(\dfrac{1}{3}\right)\right), y \in W_p^2\left(0, \dfrac{1}{3}\right) \oplus W_p^2\left(\dfrac{1}{3}, 1\right), \\ y'(0) = y'(1) = 0, y\left(\dfrac{1}{3} - 0\right) = y\left(\dfrac{1}{3} + 0\right), \end{cases}$$

and for $\hat{y} = D(L)$

$$L\hat{y} = \left(-y''; y'\left(\frac{1}{3} - 0\right) - y'\left(\frac{1}{3} + 0\right), \hat{y} \in D(L)\right).$$

Operator defined by these formula is a linear closed operator with dense definitional domain in $L_p(0,1)\oplus C$. Eigenvalues of the operator L and problem (1), (2) coincide, and $\{y_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ are eigenvectors of the operator L, where

$$\hat{y}_0=(1,m),\ \hat{y}_{i,n}=\left(y_{i,n}(x);my\left(\frac{1}{3}\right)\right),\ i=1,2;\ n\in Z^+$$

Theorem 3.1. *The system* $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ *of eigen- and associated vectors of the operator* L, *which linearized the problem (1), (2) forms basis in space* $L_p(0,1)\oplus C,\ 1<p<\infty$, *and for* $p=2$, *it forms a Riesz basis.*

Proof. Considering (6) in (7), we will get the following functional system for the head parts of the asymptotic formulas:

$$\begin{cases} u_{1,n}=\begin{cases}-\cos\left(3\pi n+\dfrac{3\pi}{2}\right)x, & x\in\left[0,\dfrac{1}{3}\right],\\ 0, & x\in\left[\dfrac{1}{3};1\right],\end{cases}\\ u_{2,n}=\begin{cases}0, & x\in\left[0,\dfrac{1}{3}\right],\\ \alpha_n\cos\left(\dfrac{3\pi n}{2}+\dfrac{3\pi}{4}\right)(1-x), & x\in\left[\dfrac{1}{3};1\right],\end{cases}\end{cases} \tag{8}$$

Where $\alpha_n=\cos\left(\frac{\pi n}{2}+\frac{\pi}{4}\right)$. If $n=4k$ and $n=4k+3$, $\alpha_n=1/\sqrt{2}$ and $n=4k+1$ and $n=4k+2$, $\alpha_n=-1/\sqrt{2}$. Then from the formulas (7) and (8) implies that, the following asymptotic relations are true:

$$\begin{cases} y_{1,n}(x)=u_{1,n}(x)+O\left(\dfrac{1}{n}\right)\\ y_{2,n}(x)=u_{2,n}(x)+O\left(\dfrac{1}{n}\right).\end{cases} \tag{9}$$

Since the operator L has compact resolvent, the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ of eigenfunctions and associated vectors is minimal in

$L_p(0,1)\oplus C$. The conjugate system $\{\hat{v}_0\}\cup\{\hat{v}_{i,n}\}_{i=1,2;n\in Z^+}$ is the eigenvectors and associated vectors of the conjugating operator L^* and is in the form $\hat{v}_0 = c_0(1;\overline{m})$, $\hat{v}_{i,n} = \left(v_{i,n}(x);\overline{m}v_{i,n}\left(\frac{1}{3}\right)\right)$, where $v_0(x)$ and $v_{i,n}(x),\ i=1,2;\ n\in Z^+$ are the eigen – and associated vectors of the conjugate problem and analogically we obtain the asymptotically formulas:

$$v_{i,n}(x)=\begin{cases} c_{i,n}\cos\dfrac{2\rho_{i,n}}{3}\cos\rho_{i,n}x, & x\in\left[0,\dfrac{1}{3}\right],\\ c_{i,n}\cos\dfrac{\rho_{i,n}}{3}\cos\rho_{i,n}(1-x), & x\in\left[\dfrac{1}{3},1\right],\end{cases}\quad i=1,2;\ n\in Z^+, \tag{10}$$

here $c_0, c_{i,n}$ are the normalized multipliers. We can easily calculate that, the c_{1n}, c_{2n} normalized multipliers hold

$$c_0=\frac{1}{1+|m|^2},\ c_{1n}=6+O\left(\frac{1}{n}\right),\ c_{2n}=6+O\left(\frac{1}{n}\right).$$

If we consider these formulas at (10) we will obtain for $v_{i,n}(x),\ i=1,2;$ $n\in Z^+$ the following formulas:

$$v_{1,n}=\begin{cases} -6\cos\left(3\pi n+\dfrac{3\pi}{2}\right)x+O\left(\dfrac{1}{n}\right) & x\in\left[0,\dfrac{1}{3}\right]\\ O\left(\dfrac{1}{n}\right), & x\in\left[\dfrac{1}{3};1\right]\end{cases} \tag{11}$$

$$v_{2,n}=\begin{cases} O\left(\dfrac{1}{n}\right), & x\in\left[0,\dfrac{1}{3}\right]\\ \alpha_n\cos\left(\dfrac{3\pi n}{2}+\dfrac{3\pi}{4}\right)(1-x)+O\left(\dfrac{1}{n}\right), & x\in\left[\dfrac{1}{3};1\right].\end{cases} \tag{12}$$

One can easily seen that the system (8) implies from the following system by the conversion

$u_{i,n}=a_{i,1}e_{1,n}+a_{i,2}e_{2,n},\ i=1,2;\ n\in Z^+$

$$\begin{cases} e_{1,n} = \begin{cases} \cos\left(3\pi n + \dfrac{3\pi}{2}\right)x, & x \in \left[0, \dfrac{1}{3}\right], \\ 0, & x \in \left[\dfrac{1}{3}, 1\right], \end{cases} \\ e_{2,n} = \begin{cases} 0, & x \in \left[0, \dfrac{1}{3}\right], \\ \alpha_n \cos\left(\dfrac{3\pi n}{2} + \dfrac{3\pi}{4}\right)(1-x), & x \in \left[\dfrac{1}{3}, 1\right], \end{cases} \end{cases} \tag{13}$$

where the numbers $a_{i,1}$ and $a_{i,2}$, $i = 1,2$ are the elements of the following matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & \alpha_n \end{pmatrix} \tag{14}$$

Note that,

$$\det A = -\alpha_n \neq 0$$

On the other hand the system $\{e_{i,n}\}_{i=1,2;n\in Z^+}$ forms a basis for $L_p(0,1)$, $1 < p < \infty$. İndeed, according to the decomposition $L_p(0,1) = L_p\left(0, \frac{1}{3}\right) \oplus L_p\left(\frac{1}{3}, 1\right)$ and since the system $\{e_{1,n}\}_{n\in Z^+}$ forms basis in $L_p\left(0, \frac{1}{3}\right)$, and the system $\{e_{2,n}\}_{n\in Z^+}$ forms basis in $L_p\left(\frac{1}{3}, 1\right)$, therefore, their combination will forms a basis in $L_p(0,1)$. If we take it into consideration and apply Theorem 2.3, then we obtain that the system $\{u_{i,n}\}_{i=1,2;n\in Z^+}$ forms basis in $L_p(0,1)$. Consider the system in $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ in $L_p(0,1) \oplus C$, where

$$\hat{u}_0 = (0,1),\ \hat{u}_{i,n} = (u_{i,n};0),\ i = 1,2;\ n \in Z^+. \tag{15}$$

It is clear that, the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ forms basis in $L_p(0,1) \oplus C$. Let us show that it also forms a q-basis, where $q = p/(p-1)$. One can easily check that the system $\{\hat{v}_0\} \cup \{\hat{\vartheta}_{i,n}\}_{i=1,2;n\in Z^+}$, which biorthogonal to it is in the following form:

$$\hat{v}_0 = \frac{1}{1+|m|^2}(1;m), \ \hat{\vartheta}_{i,n} = (\vartheta_{i,n};0), \ i=1,2; \ n \in Z^+. \qquad (16)$$

where

$$\vartheta_{1,n} = \begin{cases} -6\cos\left(3\pi n + \frac{3\pi}{2}\right)x, & x \in \left[0, \frac{1}{3}\right] \\ 0, & x \in \left[\frac{1}{3};1\right] \end{cases} \qquad (17)$$

$$\vartheta_{2,n} = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right] \\ \alpha_n \cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x), & x \in \left[\frac{1}{3};1\right]. \end{cases} \qquad (18)$$

Let $1 < p < \infty$. Then according to inequality Hausdorf-Young for trigonometric system (see [21], p.153) for each $f \in L_p(0,1)$ the inequality

$$\left(\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left|\langle f, e_{i,n}\rangle\right|^q\right)^{\frac{1}{q}} \le M\|f\|_{L_p}$$

is fulfilled, where $M > 0$ is a fixed number which does not depend on f. Taking into consideration that, the system $\{\vartheta_{i,n}\}_{i=1,2;n\in Z^+}$ implies from the system $\{e_{i,n}\}_{i=1,2;n\in Z^+}$ by conversion

$$\vartheta_{i,n} = b_{i,1}e_{1,n} + b_{i,2}e_{2,n}, \ i=1,2; \ n \in Z^+,$$

where $b_{i,1}$ and $b_{i,2}$, $i=1,2$ are the elements of the matrix

$$B = \begin{pmatrix} -6 & 0 \\ 0 & 6\alpha_n \end{pmatrix}.$$

We obtain from here that for an arbitrary $\hat{f} \in L_p(0,1) \oplus C$ the following inequality holds:

$$\left(\left|\left\langle \hat{f},\hat{\vartheta}_0\right\rangle\right|^q+\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left|\left\langle f,e_{i,n}\right\rangle\right|^q\right)^{\frac{1}{q}}\le M\left\|\hat{f}\right\|_{L_p\oplus C}$$

and implies the system $\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ is a *q*-basis in $L_p(0,1)\oplus C.$ Let's point

$$\hat{y}_{i,n}=\left(y_{i,n}(x);my_{i,n}\left(\frac{1}{3}\right)\right),\ i=1,2;\ n\in Z^+.$$

According to the formulas (7) since $y_{i,n}\left(\frac{1}{3}\right)=O\left(\frac{1}{n}\right)$, from (9) implies that the systems $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ and $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ are *p*-close,

$$\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left\|\hat{y}_{i,n}-\hat{u}_{i,n}\right\|^p<\infty..$$

Thus, all the conditions of Theorem 2.1 are fulfilled and according to this theorem the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ also forms an isomorphic basis to the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ in $L_p(0,1)\oplus C.$

Now suppose that $p>2$, then$1<q<2$. Taking into account that in this case the following inclusion is fulfilled:

$$L_p(0,1)\subset L_q(0,1)$$

then for $\hat{f}\in L_p(0,1)\oplus C,$ we obtain:

$$\left(\left|\left\langle \hat{f},\hat{\vartheta}_0\right\rangle\right|^p+\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left|\left\langle f,e_{i,n}\right\rangle\right|^p\right)^{\frac{1}{p}}\le M\left\|\hat{f}\right\|_{L_q\oplus C}\le M\left\|\hat{f}\right\|_{L_p\oplus C}.$$

This implies that the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ forms a *p*-basis in $L_p(0,1)\oplus C.$ Besides, the systems $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ and $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ are *q*-close in $L_p(0,1)\oplus C$:

$$\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left\|\hat{y}_{i,n}-\hat{u}_{i,n}\right\|_{L_p\oplus C}^q<\infty.$$

According to the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ is minimal in $L_p(0,1) \oplus C$ and again applying the Theorem 2.1, we obtain that it is a basis in $L_p(0,1) \oplus C$.isomorphic to $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$.

In the case $p = 2$ according to the Theorem 2.4 the system $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ forms a Riesz basis in $L_2(0,1) \oplus C$. Besides the systems $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ and $\{\hat{u}_0\} \cup \{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ are square-close,

$$\sum_{i=1}^{2} \sum_{n=0}^{\infty} \left\| \hat{y}_{i,n} - \hat{u}_{i,n} \right\|^2 < \infty.$$

and according to Theorem 2.1 the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ forms a Riesz basis in $L_2(0,1) \oplus C$ and this completes the proof of the theorem.

Now consider the basicity $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}^{\infty}$ of the system of eigenfunctions and associated functions of the problem (1), (2) in $L_p(0,1)$. Applying the Theorem 2 and 6, we obtain the truth of the following theorem.

Theorem 3.2. In order the system $\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+, n\neq n_0}^{\infty}$ of eigenfunctions and associated functions of the problem (1),(2) forms a basis in $L_p(0,1), 1 < p < \infty$, and for $p = 2$ forms a Riesz basis, after eliminate any function $y_{i,n_0}(x)$ it is necessary and sufficient that the corresponding function $v_{i,n_0}(x)$ of the biorthogonal system satisfy the condition $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0$. If $v_{i,n_0}\left(\frac{1}{3}\right) = 0$ then after the eliminating function $y_{i,n_0}(x)$ from the system, obtaining system does not form basis in $L_p(0,1)$, moreover in this case it is not complete and not minimal in this space.

In (6) and (7) the parametr m which included in the problem (1), (2), generally speaking is a complex number. But in some particular cases it is possible to refine the root subspaces of the operator L. So, if $m > 0$, then the operator L linearized of the problem (1), (2) is a self - adjoint operator in $L_2 \oplus C$, and in this case all the eigenvalues are simple and for each eigenvalue there corresponds only one eigenvector. If $m > 0,$ then the operator L is a J -self-adjoint operator in $L_2 \oplus C,$ and in this case applying the results of [22,23], we obtain that all eigenvalues are real and simple, with the exception of, may be either one pair of complex conjugate simple eigenvalues or one non-simple real value. In the case of a complex value m the operator L has an infinite number of complex eigenvalues that are asymptotically simple and, consequently, the operator L can have a finite number of associated vectors. If there are associated vectors, they are determined up to a linear combination with the corresponding eigenvector. Therefore depending on the choice of the coefficients of the linear combination there are associated vectors satisfying the condition $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0,$ and there are also associated vectors not satisfying this condition.

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