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ON A BOUNDARY VALUE PROBLEM FOR DIFFUSION OPERATORS WITH NON-SEPARATED BOUNDARY CONDITIONS CONTAINING THE SPECTRAL PARAMETER

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Abstract

In this paper a boundary value problem is considered generated by the diffusion equation and non-separated boundary conditions, one of which contains a spectral parameter. Some spectral properties of the boundary value problem are studied. It is proved that the eigenvalues are real and nonzero and that there are no associated functions to the eigenfunctions.

Keywords: Non-separated boundary conditions, eigenvalues, diffusion equation, associated functions to the eigenfunctions.

Mathematics Subject Classification *(2020): 34A55; 34B24; 34L05; 47E05*

1. Introduction

When solving some applied problems of mathematical physics [1], there arises a boundary value problem generated on the interval $[0,\pi]$ by the differential diffusion equation

$$y'' + \left[\lambda^2 - 2\lambda p(x) - q(x)\right] y = 0 \tag{1}$$

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and non-separated boundary conditions of the form

$$y'(0)+(\alpha\lambda+\beta)y(0)+\omega y(\pi)=0,$$
$$y'(\pi)+\gamma\, y(\pi)-\overline{\omega}y(0)=0, \tag{2}$$

where $p(x)\in W_2^1[0,\pi]$, $q(x)\in L_2[0,\pi]$ are real functions, α, β, γ are arbitrary real numbers, and ω is an arbitrary complex number. We will denote this problem by $L(\omega,\alpha,\beta)$. For $\omega=0$ the boundary conditions turn out to be separated (i.e. the boundary conditions are given separately at the points 0 and π). The papers of many authors (see [1]-[12] and references therein), have been devoted to investigation of spectral properties of second-order differential operators with a spectral parameter in the boundary condition. Periodic, anti-periodic, quasi-periodic, generalized periodic problems, in general, boundary value problems for differential equation (1) and differential pencil of Sturm – Liouville equations at non-separated boundary conditions (i.e. when boundary forms contain combination of the values of the sought function on the ends of the segment) play an important role in many physical and technical applications. Such problems with boundary conditions without a spectral parameter were studied in the papers [7], [8], [11], [12].

In the present paper a boundary value problem (1), (2) is considered in the case $\alpha\neq 0,\ \omega\neq 0,$ i.e. when one of the non-separated boundary conditions contains a spectral parameter. Some spectral properties of the boundary value problem $L(\omega,\alpha,\beta)$ are studied. It is proved that the eigenvalues are real and nonzero and that there are no associated functions to the eigenfunctions.

2. Spectral properties of the problem $L(\omega,\alpha,\beta)$

In this section we will assume everywhere that the following condition is satisfied: for all functions $y(x)\in W_2^2[0,\pi],\ y(x)\neq 0$ satisfying conditions (1.2), the following inequality holds:

$$K=\gamma|y(\pi)|^2-2\,\mathrm{Re}\left[\omega\overline{y(0)}y(\pi)\right]-\beta|y(0)|^2+\int_0^\pi\left\{|y'(x)|^2+q(x)|y(x)|^2\right\}dx>0 \tag{3}$$

Note that inequality (3) is certainly satisfied, if

$$\beta\leq 0,\ \gamma\geq 0,\ |\omega|\leq\sqrt{|\beta|\gamma}\ ,\ q(x)>0\,.$$

Definition. A complex number λ_0 is called an eigenvalue of a boundary value problem

$L(\omega,\alpha,\beta)$, if the equation (1.1) has a nontrivial solution $y_0(x)$ for $\lambda=\lambda_0$ that satisfies boundary conditions (1.2); in this case $y_0(x)$ is called the eigenfunction of the problem $L(\omega,\alpha,\beta)$ which corresponds to the eigenvalue λ_0. The set of eigenvalues is called the spectrum of the problem $L(\omega,\alpha,\beta)$. Functions

$$y_1(x),\ y_2(x),\ldots,y_r(x)$$

are called associated functions of the eigenfunction $y_0(x)$, if these functions have an absolutely continuous derivative and satisfy the differential equations

$$y_j''(x)+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]y_j(x)+[2\lambda_0-2p(x)]y_{j-1}(x)+y_{j-2}(x)=0$$

and boundary conditions

$$(\alpha\lambda_0+\beta)y_j(0)+y_j'(0)+\omega\, y_j(\pi)+\alpha y_{j-1}(0)=0,$$
$$-\overline{\omega}y_j'(0)+\gamma y_j(\pi)+y_j'(\pi)=0,$$
$$j=1,2,3,\ldots,r \quad (y_{-1}(x)\equiv 0).$$

Below we will give some auxiliary statements, which in particular cases were established in papers [9], [10].

Lemma1. *The eigenvalues of the boundary value problem* $L(\omega,\alpha,\beta)$ *are real and nonzero.*

Proof. Let λ be the eigenvalue of the problem $L(\omega,\alpha,\beta)$ and $y(x)$ be the corresponding eigenfunction. We multiply both parts of equation (1) by the function $\overline{y(x)}$ and integrate the resulting equality along the segment $[0,\pi]$ with respect to the variable x:

$$\int_0^\pi y''(x)\cdot\overline{y(x)}dx+\lambda^2\int_0^\pi|y(x)|^2dx-2\lambda\int_0^\pi p(x)|y(x)|^2dx-\int_0^\pi q(x)|y(x)|^2dx=0\,. \quad (4)$$

Applying the formula of integration by parts to the first integral , we have

$$\int_0^\pi y''(x)\cdot\overline{y(x)}dx=\int_0^\pi\overline{y(x)}dy'(x)=\overline{y(x)}y'(x)\Big|_0^\pi-\int_0^\pi y'(x)d\,\overline{y(x)}=$$
$$=\overline{y(\pi)}y'(\pi)-\overline{y(0)}y'(0)-\int_0^\pi|y'(x)|^2dx.$$

According to the boundary conditions (2), we get

$$\int_0^\pi y''(x)\cdot\overline{y(x)}dx=(\alpha\lambda+\beta)|y(0)|^2-\gamma|y(\pi)|^2+2\,\mathrm{Re}\left[\omega\overline{y(0)}y(\pi)\right]-\int_0^\pi|y'(x)|^2dx. \quad (5)$$

Substituting (5) into (4) and taking into account formula (3), we obtain

$$\lambda^2\int_0^\pi |y(x)|^2 dx - \lambda\left[2\int_0^\pi p(x)|y(x)|^2 dx - \alpha|y(0)|^2\right] - K = 0 \quad (6)$$

Solving equation (6), we obtain

$$\lambda = \frac{2\int_0^\pi p(x)|y(x)|^2 dx - \alpha|y(0)|^2 \pm \sqrt{\left[2\int_0^\pi p(x)|y(x)|^2 dx - \alpha|y(0)|^2\right]^2 + 4K\int_0^\pi |y(x)|^2 dx}}{2\int_0^\pi |y(x)|^2 dx}. \quad (7)$$

From relations (3), (7) and from the fact that $p(x)$ is a real function, the validity of the theorem follows.

The lemma is proved.

Lemma 2. *If $y(x)$ is the eigenfunction of the problem $L(\omega,\alpha,\beta)$ corresponding to the eigenvalue λ, then*

$$2\int_0^\pi [\lambda - p(x)]|y(x)|^2 dx + \alpha|y(0)|^2 \neq 0 \quad (8)$$

Moreover, the sign of the left side of this relation coincides with the sign of λ.

Proof. From equation (6) the following relation is easily obtained:

$$2\int_0^\pi [\lambda - p(x)]|y(x)|^2 dx + \alpha|y(0)|^2 = \pm\sqrt{\left[2\int_0^\pi p(x)|y(x)|^2 dx - \alpha|y(0)|^2\right]^2 + 4K\int_0^\pi |y(x)|^2 dx}$$

According to the condition (3), since $K > 0$, the right side of this equation is different from zero. Therefore, the left side is also non-zero, that is, relation (8) is true. It is also clear from (7), that for $\lambda > 0$ there must be a ”+” sign in front of the root, and for $\lambda < 0$ there must be a ”-” sign. From here we find that the sign of the expression $\int_0^\pi [\lambda - p(x)]|y(x)|^2 dx + \alpha|y(0)|^2$ coincides with the sign of λ.

The lemma is proved.

Now we are prepared to prove the main claim.

Theorem. *The boundary value problem $L(\omega,\alpha,\beta)$ has no associated functions of the eigenfunctions.*

Proof. Let us assume the opposite. Let us suppose there is an associated function $y_1(x)$ of the eigenfunction $y_0(x)$ of problem $L(\omega,\alpha,\beta)$, which corresponds to the eigenvalue λ_0. Then the following equalities hold:

$$y_0''(x)+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]y_0(x)=0, \tag{9}$$

$$y_1''(x)+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]y_1(x)+\left[2\lambda_0-2p(x)\right]y_0(x)=0. \tag{10}$$

Let us pass in equality (9) to the complex conjugate and then multiply the resulting equality by $y_1(x)$, and multiply relation (10) by $\overline{y_0(x)}$:

$$\overline{y_0''(x)}y_1(x)+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]\overline{y_0(x)}y_1(x)=0,$$

$$y_1''(x)\overline{y_0(x)}+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]\overline{y_0(x)}y_1(x)+\left[2\lambda_0-2p(x)\right]y_0(x)\overline{y_0(x)}=0.$$

subtract the second result from the first:

$$2\left[\lambda_0-p(x)\right]y_0(x)\overline{y_0(x)}=\overline{y_0''(x)}y_1(x)-y_1''(x)\overline{y_0(x)}.$$

The last equality can be rewritten as

$$2\left[\lambda_0-p(x)\right]y_0(x)\overline{y_0(x)}=\frac{d}{dx}\left[\overline{y_0'(x)}y_1(x)-y_1'(x)\overline{y_0(x)}\right]. \tag{11}$$

After integrating the relation (11) over x from 0 to π, we get

$$2\left[\lambda_0-p(x)\right]\int_0^\pi \left|y_0(x)\right|^2 dx=\left[\overline{y_0'(x)}y_1(x)-y_1'(x)\overline{y_0(x)}\right]\Big|_0^\pi=$$

$$=\overline{y_0'(\pi)}y_1(\pi)-\overline{y_0'(o)}y_1(0)-y_1'(\pi)\overline{y_0(\pi)}+y_1'(0)\overline{y_0(0)}. \tag{12}$$

Taking into account that the functions $y_0(x)$ and $y_1(x)$ included in equality (12) satisfy the boundary conditions

$$(\alpha\lambda_0+\beta)y_0(0)+y_0'(0)+\omega y_0(\pi)=0,$$
$$-\overline{\omega}y_0(0)+\gamma y_0(\pi)+y_0'(\pi)=0,$$

and

$$(\alpha\lambda_0+\beta)y_1(0)+y_1'(0)+\omega y_1(\pi)+\alpha y_0(0)=0,$$
$$-\overline{\omega}y_1(0)+\gamma y_1(\pi)+y_1'(\pi)=0,$$

we receive

$$2\left[\lambda_0-p(x)\right]\int_0^\pi \left|y_0(x)\right|^2 dx=\left(\omega\overline{y_0(0)}-\gamma\overline{y_0(\pi)}\right)y_1(\pi)+\left[(\alpha\lambda_0+\beta)\overline{y_0(0)}+\right.$$

$$\left.+\overline{\overline{\omega}y_0(\pi)}\right]y_1(0)-\left[\overline{\omega}y_1(0)-\gamma y_1(\pi)\right]\overline{y_0(\pi)}-\left[(\alpha\lambda_0+\beta)y_1(0)+\omega y_1(\pi)+\right.$$

$$+\alpha y_0(0)]\overline{y_0(0)} = \omega\overline{y_0(0)}y_1(\pi) - \gamma\overline{y_0(\pi)}y_1(\pi) + \alpha\overline{y_0(0)}y_1(0 +$$
$$+\overline{\omega}\overline{y_0(\pi)}y_1(0) + \beta y_1(0)\overline{y_0(0)} - \overline{\omega}y_1(0)\overline{y_0(\pi)} + \gamma y_1(\pi)\overline{y_0(\pi)} -$$
$$-\alpha\lambda_0 y_1(0)\overline{y_0(0)} - \beta y_1(0)\overline{y_0(0)} - \omega y_1(\pi)\overline{y_0(0)} -$$
$$-\alpha y_0(0)\overline{y_0(0)} = -\alpha|y_0(0)|^2.$$

From this we get

$$2\int_0^{\pi}[\lambda_0 - p(x)]|y_0(x)|^2 dx + \alpha|y_0(0)|^2 = 0,$$

which contradicts the inequality (8).

The theorem is proved.

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