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ON A SOLVABILITY OF THE NONLINEAR INVERSE BOUNDARY VALUE PROBLEM FOR PSEUDO HYPERBOLIC EQUATION OF THE FOURTH ORDER

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Abstract

We study the classical solution of the nonlinear inverse boundary value problem for for pseudo hyperbolic equation of the fourth order The essence of the problem is that it is required together with the solution to determine the unknown coefficient. The problem is considered in a rectangular area. To solve the considered problem, the transition from the original inverse problem to some auxiliary inverse problem is carried out. The existence and uniqueness of a solution to the auxiliary problem are proved with the help of contracted mappings. Then the transition to the original inverse problem is made, as a result, a conclusion is made about the solvability of the original inverse problem.

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1. Introduction

There are many cases where the needs of the practice bring about the problems

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of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2, 3], V.K. Ivanov [4] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [5]. In this paper, following [6,7], we prove existence and uniqueness of the solution to an inverse boundary value problem for pseudo hyperbolic equation of the fourth order .

Formulation of the problem and its equivalent form

Let $D_T=\{(x,t):\ 0\le x\le 1,\ \ 0\le t\le T\}$. and $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)\ (i=1,2)$ are given functions defined for $x\in[0,1],\ t\in[0,T]$. Consider the following inverse problem: to find a triple $\{u(x,t),\ a(t),b(t)\}$ of the functions $u(x,t)$, $a(t)$, $b(t)$ satisfying the equation

$$u_{tt}(x,t)-u_{ttxx}(x,t)+u_{xxxx}(x,t)=$$
$$=a(t)u(x,t)+b(t)u_t(x,t)+f(x,t) \tag{1}$$

with initial

$$u(x,0)=\varphi(x),\ \ u_t(x,0)=\psi(x)\quad (0\le x\le 1), \tag{2}$$

and boundary conditions

$$u(0,t)=u_x(1,t)=u_{xx}(0,t)=u_{xxx}(1,t)=0\quad (0\le t\le T) \tag{3}$$

and with additional conditions

$$u(x_i,t)=h_i(t)\qquad (0<x_i<1, i=1,2; x_1\ne x_2;0\le t\le T), \tag{4}$$

Introduce the designation

$$\tilde{C}^{4,2}(D_T)=\left\{u(x,t):u(x,t)\in C^2(D_T),u_{xxxx}(x,t),u_{ttxx}(x,t\in C(D_T)\right\}$$

Definition. *A triple* $\{u(x,t),a(t),b(t)\}$ *of the functions* $u(x,t)\in C^{4,2}(D_T)$ *,*

$a(t) \in C[0,T]$ and $b(t) \in C[0,T]$ satisfying equation (1) in D_T, condition (2) in $[0,1]$ and conditions (3)-(4) in $[0,T]$ we call a classical solution to boundary value (1)-(4).

We prove the following

Теорема 1. *Let $f(x,t) \in C(D_T)$, $\varphi(x)$, $\psi(x) \in C[0,1]$, $h_i(t) \in C^2[0,T] (i=1,2)$ $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \neq 0$ $(0 \le t \le T)$ and the matching conditions*

$$\varphi(x_i) = h_i(0),\ \psi(x_i) = h_i'(0)\ .(i=1,2)$$

are satisfied. Then the problem of finding a classical solution to problem (1)-(4) is equivalent to the problem of determining the functions $u(x,t) \in C^{4,2}(D_T)$, $a(t) \in C[0,T]$ and $b(t) \in C[0,T]$ from (1)-(3) and

$$h_i''(t) - u_{ttxx}(x_i,t) + u_{xxxx}(x_i,t) =$$
$$= a(t)h_i(t) + b(t)h_i'(t) + f(x_i,t)\ (i=1,2;\ 0 \le t \le T). \tag{5}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a classical solution to problem (1)-(4). Since $h_i(t) \in C^2[0,T] (i=1,2)$, differentiating (4) two times over t we get

$$u_t(x_i,t) = h_i'(t),\ u_{tt}(x_i,t) = h_i''(t) \quad (i=1,2;\ 0 \le t \le T). \tag{6}$$

Taking $x = x_i$ in equation (1) we find

$$u_{tt}(x_i,t) - u_{ttxx}(x_i,t) + u_{xxxx}(x_i,t) =$$
$$= a(t)u(x_i,t) + b(t)u_t(x_i,t) + f(x_i,t)\ \ (i=1,2;\ 0 \le t \le T) \tag{7}$$

From this considering (4) and (6) we arrive at (5).

Now let's suppose that $\{u(x,t), a(t), b(t)\}$ is a solution of problem (1)-(3), (5). Then from (5) and (7) we get

$$\frac{d^2}{dt^2}(u(x_i,t) - h_i(t)) = a(t)(u(x_i,t) - h_i(t)) + b(t)\frac{d}{dt}(u(x_i,t) - h_i(t))$$
$$(i=1,2; 0 \le t \le T)\ . \tag{8}$$

Considering (2) and $\varphi(x_i) = h_i(0)$, $\psi(x_i) = h_i'(0)$ $.(i=1,2)$ we have

$$u(x_i,0)-h_i(0)=\varphi(x_i)-h_i(0)=0,$$

. $$u_t(x_i,0)-h_i'(0)=\psi(x_i)-h_i'(0)=0 \quad (i=1,2)\ . \tag{9}$$

From (8), taking into account (9), it is clear that condition (4) is also satisfied. The theorem is proved.

Solvability of the inverse boundary value problem

The first component $u(x,t)$ of the solution $\{u(x,t),a(t),b(t)\}$ to problem (1)-(3), (5) we seek in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x \quad \left(\lambda_k=\frac{\pi}{2}(2k-1)\right), \tag{10}$$

where

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx \ (k=1,2,...).$$

Then applying the formal Fourier scheme, from (1) and (2) we obtain

$$(1+\lambda_k^2)u_k''(t)+\lambda_k^4 u_k(t)=F_k(t;u,a,b) \ (0\le t\le T; k=1,2,...) \tag{11}$$

$$u_k(0)=\varphi_k,\ u_k'(0)=\psi_k \quad (k=1,2,...), \tag{12}$$

where

$$F_k(t;u,a,b)=a(t)u_k(t)+b(t)u_k'(t)+f_k(t) \quad ,\ f_k(t)=2\int_0^1 f(x,t)\sin\lambda_k x\,dx,$$

$$\varphi_k=2\int_0^1\varphi(x)\sin\lambda_k xdx,\quad \psi_k=2\int_0^1\psi(x)\sin\lambda_k xdx \quad (k=1,2,...).$$

Solving problem (11)-(12) we find

$$u_k(t)=\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau \ (k=1,2,...), \tag{13}$$

where

$$\beta_k^2 = \frac{\lambda_k^4}{1+\lambda_k^2} \quad (k=1,2,...) .$$

After substitution of the expression $u_k(t)$ $(k=1,2,...)$ into (10) for the determination of $u(x,t)$ we get

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \varphi_k \cos \beta_k t + \frac{1}{\beta_k} \psi_k \sin \beta_k t \right. +$$

$$\left. + \frac{1}{\beta_k (1+\lambda_k^2)} \int_0^t F_k(\tau; u, a, b) \sin \beta_k (t-\tau) d\tau \right\} \sin \lambda_k x. \qquad (14)$$

Now from (5) taking into account (10) we have

$$a(t) = [h(t)]^{-1} \left\{ (h_1''(t) - f(x_1,t))\, h_2'(t) - (h_2''(t) - f(x_2,t)) h_1'(t) + \right.$$

$$\left. + \sum_{k=1}^{\infty} (\lambda_k^2 u_k''(t) + \lambda_k^4 u_k(t))(h_2'(t) \sin \lambda_k x_1 - h_1'(t) \sin \lambda_k x_2) \right\}, \qquad (15)$$

$$b(t) = [h(t)]^{-1} \left\{ (h_2''(t) - f(x_2,t))\, h_1(t) - (h_1''(t) - f(x_1,t)) h_2(t) + \right.$$

$$\left. + \sum_{k=1}^{\infty} (\lambda_k^2 u_k''(t) + \lambda_k^4 u_k(t))(h_1(t) \sin \lambda_k x_2 - h_2(t) \sin \lambda_k x_1) \right\}. \qquad (16)$$

Consideration of (13) in (11) gives

$$\lambda_k^2 u_k''(t) + \lambda_k^4 u_k(t) = -u_k''(t) + F_k(t; u, a, b) =$$

$$= \frac{\lambda_k^4}{1+\lambda_k^2} u_k(t) + \left(1 - \frac{1}{1+\lambda_k^2}\right) F_k(t; u, a, b) =$$

$$= \frac{\lambda_k^4}{1+\lambda_k^2} u_k(t) + \frac{\lambda_k^2}{1+\lambda_k^2} F_k(t; u, a, b) = \beta_k^2 u_k(t) + \frac{\lambda_k^2}{1+\lambda_k^2} F_k(t; u, a, b) =$$

$$=\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\right.$$

$$\left.+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau\right]+\frac{\lambda_k^2}{1+\lambda_k^2}F_k(t;u,a,b),$$

$$k=1,2,\ldots,\ 0\le t\le T.$$

To obtain an equation for the second component $a(t),\ b(t)$ of the solution $\{u(x,t),a(t),b(t)\}$ we put the last relation into (15)

$$a(t)=[h(t)]^{-1}\left\{(h_1''(t)-f(x_1,t))\,h_2'(t)-(h_2''(t)-f(x_2,t))h_1'(t)+\right.$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\right.$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau+$$

$$\left.\left.+\frac{1}{\lambda_k^2}F_k(t;u,a,b)\right](h_2'(t)\sin\lambda_k x_1-h_1'(t)\sin\lambda_k x_2)\right\}, \tag{17}$$

$$b(t)=[h(t)]^{-1}\left\{(h_2''(t)-f(x_2,t))\,h_1(t)-(h_1''(t)-f(x_1,t))h_2(t)+\right.$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t+\right.$$

$$\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_k(\tau;u,a,b)\sin\beta_k(t-\tau)d\tau+$$

$$\left. + \frac{1}{\lambda_k^2} F_k(t;u,a,b) \right] (h_1(t)\sin\lambda_k x_2 - h_2(t)\sin\lambda_k x_1) \Bigg\} , \qquad (18)$$

This, solution of problem (1)-(3),(5) is reduced to the solution of system (14), (17),(18) with respect to the unknown functions $u(x,t)$, $a(t)$. and $b(t)$.

To study the problem of the uniqueness of the solution of problem (1)-(3), (5), the following lemma plays an important role.

Lemma. *If* $\{u(x,t),a(t),b(t)\}$ *is arbitrary classical solution of problem (1)-(3), (5), then the function*

$$u_k(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \ \ (k=1,2,...)$$

satisfies system (13) in $[0,T]$.

Proof. Let $\{u(x,t),a(t),b(t)\}$ be any solution to problem (1)-(3), (5). Then multiplying both sides of equation (1) by the function $2\sin\lambda_k x \ (k=1,2,...)$, integrating the obtained equality over x from 0 to 1 and using the relations

$$2\int_0^1 u_{tt}(x,t)\sin\lambda_k x dx = \frac{d^2}{dt^2}\left(2\int_0^1 u(x,t)\sin\lambda_k x dx\right) = u_k''(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{ttxx}(x,t)\sin\lambda_k x dx = -\lambda_k^2\left(2\int_0^1 u_{tt}(x,t)\sin\lambda_k x dx\right) = -\lambda_k^2 u_k''(t) \ \ (k=1,2,...),$$

$$2\int_0^1 u_{xxxx}(x,t)\sin\lambda_k x dx = \lambda_k^4\left(2\int_0^1 u(x,t)\sin\lambda_k x dx\right) = \lambda_k^4 u_k(t) \ \ (k=1,2,...),$$

we obtain that equation (11) is satisfied.

Similarly, the fulfilment of (12) is obtained from (2). Thus $u_k(t)\ (k=1,2,...)$ is a solution to problem (11), (12).

As immediately follows from this the function $u_k(t)\ (k=1,2,...)$ satisfies to system (13) on $[0,T]$. Lemma is proved.

This lemma implies the validity of the following

Consequence. *Let system (14), (17),(18) have a unique solution. Then problem (1)-(3), (5) cannot have more than one solution, i.e. if problem (1)-(3), (5) has a solution, then it is unique.*

Now, in order to study problem (1)-(3), (5) consider the following spaces.

1. Denote by $B_{2,T}^{5,3}$ [8] the set of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\lambda_k x \ \left(\lambda_k=\frac{\pi}{2}(2k-1)\right),$$

Defined on D_T, where each of the functions $u_k(t)\in C^1[0,T](k=1,2,\dots)$ and

$$J_T(u)\equiv\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this space is defined as

$$\|u(x,t)\|_{B_{2,T}^{5,3}}=J(u).$$

2. By $E_T^{5,3}$ we denote the space of the vector functions $\{u(x,t),a(t),b(t)\}$ such that $u(x,t)\in B_{2,T}^{5,3}$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$ and equip this space by the norm

$$\|z\|_{E_T^{5,3}}=\|u(x,t)\|_{B_{2,T}^{5,3}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

Clearly, $B_{2,T}^{5,3}$ and $E_T^{5,3}$ are Banach spaces.

Now we consider in $E_T^{5,3}$ the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\},$$

where

$$\Phi_1(u,a,b)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,\quad \Phi_2(u,a,b)=\tilde{a}(t),\Phi_3(u,a,b)=\tilde{b}(t),$$

$\tilde{u}_k(t)$ ($k=1,2,\ldots$), $\tilde{a}(t)$ and $\tilde{b}(t)$ are the right hand sides of (13) and (17),(18) correspondingly.

Obviously

$$\frac{1}{\sqrt{2}}\lambda_k < \beta_k < \lambda_k\,,\ \frac{1}{\lambda_k} < \frac{1}{\beta_k} < \frac{\sqrt{2}}{\lambda_k}\ .$$

Then we have

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_k|)^2\right)^{\frac{1}{2}} + \sqrt{10}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{10T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{10}\ T\|a(t)\|_{C[0,T]}\times$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \sqrt{10}\ T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \qquad (19)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_k|)^2\right)^{\frac{1}{2}} + \sqrt{5}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{5}\ T\|a(t)\|_{C[0,T]}\times$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \sqrt{5}\ T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \qquad (20)$$

$$\|\tilde{a}(t)\|_{C[0,T]} =$$

$$= \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t) - f(x_1,t))\,h_2'(t) - (h_2''(t) - f(x_2,t))h_1'(t)\right\|_{C[0,T]} +\right.$$

$$+2\left\||h_2'(t)| + |h_1'(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_k|)^2\right)^{\frac{1}{2}} + \sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right) +\right.$$

$$+\sqrt{2T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{2}\,T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{2}\,T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}.+\left(\sum_{k=1}^{\infty}(\lambda_k^2\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \qquad (21)$$

$$\|\tilde{b}(t)\|_{C[0,T]}=$$

$$=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|(h_2''(t)-f(x_2,t))\,h_1(t)-(h_1''(t)-f(x_1,t))h_2(t)\right\|_{C[0,T]}+$$

$$+2\left\||h_2(t)|+|h_1(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\Bigg[\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_k|)^2\right)^{\frac{1}{2}}+\sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)+$$

$$+\sqrt{2T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\sqrt{2}\,T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{2}\,T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}.+\left(\sum_{k=1}^{\infty}(\lambda_k^2\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \qquad (22)$$

where

$$\tilde{u}_k'(t)=-\beta_k\varphi_k\sin\beta_k t+\psi_k\sin\beta_k t+$$

$$+\frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a,b)\cos\beta_k(t-\tau)d\tau \ (k=1,2,...) .$$

Assume that the data of problem (1)-(3), (5) satisfy the following conditions:

1. $\varphi(x)\in C^4[0,1],\ \varphi^{(5)}(x)\in L_2(0,1),\ \varphi(0)=\varphi'(1)=\varphi''(0)=\varphi'''(1)=\varphi^{(4)}(0)=0'$
2. $\psi(x)\in C^2[0,1],\ \psi^{(3)}(x)\in L_2(0,1),\ \psi(0)=\psi'(1)=\psi''(0)=0$.
3. $f(x,t),\ f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ f(0,t)=f_x(1,t)=0\ (0\le t\le T)$.
4. $h_i(t)\in C^2[0,T](i=1,2)$, $h(t)\equiv h_1(t)h_2'(t)-h_2(t)h_1'(t)\neq 0\ (0\le t\le T)$.

Then from (19)-(22) we have

$$\|\tilde u(x,t)\|_{B_{2,T}^{5,3}}\le A_1(T)+B_1(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,3}}, \qquad (23)$$

$$\|\tilde a(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,3}}, \qquad (24)$$

$$\|\tilde b(t)\|_{C[0,T]}\le A_3(T)+B_3(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,3}} \qquad (25)$$

where

$$A_1(T)=2\sqrt5\|\varphi^{(5)}(x)\|_{L_2(0,1)}+\sqrt5(1+\sqrt2)\|\psi^{(3)}(x)\|_{L_2(0,1)}+$$
$$+\sqrt{5T}(1+\sqrt2)\|f_{xx}(x,t)\|_{L_2(D_T)},\ B_1(T)=\sqrt5(1+\sqrt2)T ,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t)-f(x_1,t))\,h_2'(t)-(h_2''(t)-f(x_2,t))h_1'(t)\right\|_{C[0,T]}+\right.$$
$$+2\left\||h_2'(t)|+|h_1'(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac12}\left[\|\varphi^{(5)}(x)\|_{L_2(0,1)}+\right.$$
$$\left.\left.+\sqrt2\|\psi^{(3)}(x)\|_{L_2(0,1)}+\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}+\left\|\|f_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right]\right\},$$

$$B_2(T) = 2\left\| \left[h(t) \right]^{-1} \right\|_{C[0,T]} \left\| |h_2'(t)| + |h_1'(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (T+1),$$

$$A_3(T) = \left\| \left[h(t) \right]^{-1} \right\|_{C[0,T]} \left\{ \left\| (h_2''(t) - f(x_2,t))\, h_1(t) - (h_1''(t) - f(x_1,t)) h_2(t) \right\|_{C[0,T]} + \right.$$

$$+ 2\left\| |h_2(t)| + |h_1(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\left\| \varphi^{(5)}(x) \right\|_{L_2(0,1)} + \right.$$

$$\left. \left. + \sqrt{2} \left\| \psi^{(3)}(x) \right\|_{L_2(0,1)} + \sqrt{3T} \| f_{xx}(x,t) \|_{L_2(D_T)} + \left\| \| f_{xx}(x,t) \|_{C[0,T]} \right\|_{L_2(0.1)} \right] \right\},$$

$$B_3(T) = 2\left\| \left[h(t) \right]^{-1} \right\|_{C[0,T]} \left\| |h_2(t)| + |h_1(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (T+1),$$

From inequalities (23)-(25) we conclude

$$\left\| \tilde{u}(x,t) \right\|_{B_{2,T}^{5,3}} + \left\| \tilde{a}(t) \right\|_{C[0,T]} + \left\| \tilde{b}(t) \right\|_{C[0,T]} \leq$$

$$\leq A(T) + B(T) \left(\left\| a(t) \right\|_{C[0,T]} + \left\| b(t) \right\|_{C[0,T]} \right) \left\| u(x,t) \right\|_{B_{2,T}^{5,3}}, \qquad (26)$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T).$$

So, we can prove the following theorem:

Theorem 2. *Let conditions 1-4 be satisfied and*

$$(A(T) + 2)^2 B(T) < 1. \qquad (27)$$

The problem (1)-(3),(5) has a unique solution in the ball $K = K_R(\|z\|_{E_T^{5,4}} \leq R = A(T) + 2)$ *of the space* $E_T^{5,3}$.

Remark. Inequality (27) is satisfied for sufficiently small values of

$$T + \left\| \left[h(t) \right]^{-1} \right\|_{C[0,T]}.$$

Proof. In the space $E_T^{5,3}$ consider the equation

$$z = \Phi z, \qquad (28)$$

where $z=\{u,a,b\}$, the components $\Phi_i(u,a,b)$ $(i=1,2,3)$ of the operator $\Phi(u,a,b)$ are defined by the right hand sides of equations (14),(17) and (18) .

Consider the operator $\Phi(u,a,b)$ in the ball $K=K_R$ from $E_T^{5,4}$. Similarly to (22) we obtain that the estimations

$$\|\Phi z\|_{E_T^{5,3}} \leq A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,3}}, \tag{29}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,3}} \leq \leq B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|b_1(t)-b_2(t)\|_{C[0,T]}+\right.$$

$$\left.+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{5,3}}\right) \tag{30}$$

for the arbitrary $z,z_1,z_2\in K_R$. Then, from estimates (29), (30), taking into account (27), it follows that the operator Φ acts in the ball and is contractive. Therefore in the ball $K=K_R$ the operator Φ has a single fixed point $\{u,a,b\}$ which is a unique solution to equation (28) in the ball $K=K_R$, i.e. $\{u,a,b\}$ is a unique solution to system (14),(17) and (18) in the ball $K=K_R$.

The function $u(x,t)$ as an element of the space $B_{2,T}^{5,3}$ has continuous derivatives $u(x,t),\, u_x(x,t),\, u_{xx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t)$, $u_t(x,t),\, u_{tx}(x,t),$ $u_{txx}(x,t)$ in D_T.

As one can easily see from

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{2}\left\|\|f_x(x,t)+a(t)u_x(x,t)+b(t)u_{tx}(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

It implies that $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to check that equation (1) and conditions (2), (3) and (5) are satisfied in the usual sense. Therefore, $\{u(x,t),a(t),b(t)\}$ is a solution to problem (1)-(3), (5), and, by virtue of the corollary of Lemma 1, it is unique in the ball

$K = K_R$.

The theorem is proved.

Using Theorem 1, we prove the following

Theorem 3. Let all conditions of Theorem 2 be satisfied and

$$\varphi(x_i) = h_i(0),\ \psi(x_i) = h_i'(0)\ .(i = 1,2).$$

The problem (1)-(4) has unique classical solution in the ball $K = K_R (\|z\|_{E_T^{5,3}} \le R = A(T) + 2)$ from $E_T^{5,3}$.

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