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ON GLOBAL CONTINUA OF NONTRIVIAL SOLUTIONS OF NONLINEAR DIRAC PROBLEMS

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Abstract

In this paper we consider global bifurcation from zero and infinity of nontrivial solutions of some nonlinear Dirac problems. We show the existence of two families of global continua of nontrivial solutions of this problem emanating from bifurcation points with respect to the line of trivial solutions that contain asymptotic bifurcation points and are contained in classes of vector-functions with fixed oscillation count.

Keywords: nonlinear Dirac problem, bifurcation from zero, bifurcation from infinity, global continua
Mathematics Subject Classification *(2020): 34A30, 34B15, 34C10, 34K29, 47J10, 47J15*

1. Introduction

In this paper we consider the following nonlinear Dirac problem

$$Bw'(x) - P(x)w(x) = \lambda w(x) + g(x, w(x)),\ x \in (0, \pi), \tag{1}$$

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$$U_1(w) = (\cos\alpha, \sin\alpha)\, w(0) = 0, \tag{2}$$

$$U_2(w) = (\cos\beta, \sin\beta)\, w(\pi) = 0, \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},\ w(x) = \begin{pmatrix} u(x) \\ \vartheta(x) \end{pmatrix},$$

$\lambda \in R$ is a spectral parameter, $p(x)$ and $r(x)$ are real-valued continuous functions on $[0, \pi]$, α and β are real constants such that $0 \le \alpha, \beta < \pi$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, where $g_1(x)$ and $g_2(x)$ are real-valued continuous functions on $[0, \pi]$. Moreover, the vector-function g satisfies the following conditions:

$$g(x, w) = o(|w|) \quad \text{as} \quad |w| \to 0 \tag{4}$$

and

$$g(x, w) = o(|w|) \quad \text{as} \quad |w| \to \infty, \tag{5}$$

uniformly in $x \in [0, \pi]$.

The Dirac equation, as a relativistic wave equation, describes the motion of particles with spin 1/2, such as electrons, positrons, protons, neutrons under the influence of external electromagnetic fields (see [11]). It should be noted that nonlinear Dirac equations were proposed to model the self-interaction of such particles and other phenomena (see, for example, [4-11]).

Problem (1)-(3) under condition (4) was considered in [3], where it was shown that there exist unbounded continua of solutions branching off from the points of the line of trivial solutions (the first components of which are the eigenvalues of the linear eigenvalue problem obtained from (1)-(3) by substituting $g \equiv 0$) and contained in classes of functions possessing oscillatory properties of this linear problem.

In the case where condition (5) is satisfied, problem (1)-(3) was considered in [1], where it was proved that there exist global continua of solutions bifurcating from the points of the line $R \times \{\infty\}$ (the first components of which are the eigenvalues of the linear problem) and contained in classes of vector-functions possessing oscillatory properties of eigenvector-functions of the linear problem in the neighborhoods of these points. Moreover, these continua either contain other bifurcation points, or intersect the line $R\times\{0\}$, or have unbounded projections onto $R\times\{0\}$.

In this paper, we consider problem (1)-(3) when both conditions (4) and (5) are satisfied. In this case, we show that the global continua branching from the line $R\times\{\infty\}$ are also contained in the classes of vector-functions possessing oscillatory properties of eigenvector-functions of the linear problem and, therefore, do not intersect other asymptotic bifurcation points. Then we prove that the projections onto the line $R\times\{0\}$ of the continua branching from zero and from infinity are bounded and, therefore, these continua coincide.

2. Preliminary

By $B.C.$ we denote the set of functions which satisfy boundary conditions (2) and (3). Let E be the Banach space $C([0,\pi];R^2)\bigcap B.C.$ with the usual norm

$$\| w\| = \max_{x\in[0,\pi]} |u(x)| + \max_{x\in[0,\pi]} |\vartheta(x)|.$$

We define a set S in space E as follows:

$$S=\{w\in E \mid |u(x)|+|\vartheta(x)|>0,\ x\in[0,\pi]\}.$$

For each $w\in E$ we define $\theta(w,x)$ to be the continuous function on $[0,\pi]$ satisfying

$$\cot\theta(w,x)=\frac{u(x)}{\vartheta(x)},\ \theta(w,0)=-\alpha. \tag{6}$$

We consider the linear eigenvalue problem

$$\begin{cases}\ell(w)(x)=\lambda w(x),\ x\in(0,\pi),\\ U(w)=\tilde{0},\end{cases} \tag{7}$$

where

$$U(w)=\begin{pmatrix}U_1(w)\\ U_2(w)\end{pmatrix} \text{ and } \tilde{0}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

which obtained from (1)-(3) by setting $g\equiv 0$. By [2, Theorem 3.1] the eigenvalues $\lambda_k, k\in Z,$ of problem (7) are real, simple and can be numbered in ascending order on the real axis as follows

$$. . . < \lambda_{-k} < . . . < \lambda_{-1} < \lambda_0 < \lambda_1 < . . . < \lambda_k < . . . ,$$

so that the corresponding angular function $\theta(w_k, x)$ at $x = 0$ and $x = \pi$ will satisfy the following relations at $x = 0$ and $x = \pi$ will satisfy the following relations

$$\theta(w_k, 0) = -\alpha \quad \text{and} \quad \theta(w_k, \pi) = -\beta + k\pi, \tag{8}$$

where $w_k(x)$ is the eigenvector-function corresponding to the eigenvalue λ_k.

For each $k \in \mathrm{Z}$ and each $\nu \in \{+, -\}$ let S_k^ν the set of vector-functions $w \in S$ which satisfy the following conditions:

(i) $\theta(w, \pi) = -\beta + k\pi;$

(ii) if $k > 0$ or $k = 0,\ \alpha \geq \beta,$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k < 0$ or $k = 0,\ \alpha < \beta,$ then for fixed w, as x increases the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above;

(iii) the function $\nu\, u(x)$ is positive in a deleted neighborhood of the point $x = 0.$

Let $S_k = S_k^+ \cup S_k^-$. Note that $S_k^+,\ S_k^+$ and S_k are disjoint and open sets in E. Moreover, if $w \in \partial S_k^\nu$, then there exists $\xi \in [0, \pi]$ such that $|w(\xi)| = |u(\xi)| + |\vartheta(\xi)| = 0$ (see [2, 3]).

Lemma 1 [3, Lemma 2.8]. *If $(\lambda, w) \in R \times E$ is a solution of problem (1)-(3) such that $w \in \partial S_k^\nu,\ k \in \mathrm{Z},\ \nu \in \{+, -\},$ then $w \equiv \tilde{0}$.*

Since the function g satisfies the condition (5) and (6) it follows from [3, Theorem 3.1] and [1, Theorem 4.1] we have the following results.

Theorem 1 [3, Theorem 3.1]. *For each $k \in \mathrm{Z}$ and each $\nu \in \{+, -\}$ there exists a continuum C_k^ν of nontrivial solutions of problem (1)-(3) which contain $(\lambda_k, \tilde{0})$, contained in $(R \times \hat{S}_k^\nu) \cup \{(\lambda_k, \tilde{0})\}$ and is unbounded in $R \times E$* (in this case either (i) C_k^ν *meets* $R \times \{\infty\}$ *for some* $\lambda \in R,$ or (ii) *the projection* $P_{R \times \{\tilde{0}\}}(C_k^\nu)$ *of* C_k^ν *onto* $R \times \{\tilde{0}\}$ *is unbounded).*

Theorem 2 [1, Theorem 4.1]. *For each $k \in \mathrm{Z}$ and each $\nu \in \{+, -\}$ there exists a* *continuum D_k^ν of nontrivial solutions of problem (1)-(3) which meet (λ_k, ∞) with respect to the set $R\times S_k^\nu$ and for this set at least one of the following holds:* (i) D_k^ν *meets (λ'_k, ∞) with respect to the set $R\times S_{k'}^{\nu'}$ for some $(k',\nu')\neq(k,\nu)$;* (ii) D_k^ν *meets $R\times\{\tilde{0}\}$ for some $\lambda\in R$;* (iii) *the projection $P_{R\times\{\tilde{0}\}}(D_k^\nu)$ of D_k^ν onto $R\times\{\tilde{0}\}$ is unbounded.*

3. The connection between global continua bifurcating from zero and from infinity

In this section we will find the connection between the continua C_k^ν and D_k^ν for each $k\in Z$ and each $\nu\in\{+,-\}$. To do this, we first prove the following lemma.

Lemma 2. For each $k\in Z$ and each $\nu\in\{+,-\}$,

$$D_k^\nu\setminus\{(\lambda_k,\infty)\}\subset R\times S_k^\nu.$$

Proof. Let $C\subset R\times E$ be the set of nontrivial solutions of problem (1)-(3). Then It follows from Lemma 1 that

$$C\bigcap(R\times\partial S_k^\nu)=\varnothing \text{ for each } k\in Z \text{ and each } \nu\in\{+,-\}.$$

Consequently, the sets

$$C\bigcap(R\times S_k^\nu) \text{ and } C\setminus(R\times S_k^\nu)$$

are mutually separated in the space $R\times E,$ whence, by [12, Corollary 26.6], implies that any component of the set C must be a subset of $C\bigcap(R\times S_k^\nu)$ or $C\setminus(R\times S_k^\nu)$. Since, by Theorem 2, the relation

$$(D_k^\nu\setminus\{(\lambda_k,\infty)\})\bigcap(R\times S_k^\nu)\neq\varnothing$$

holds, it follows that

$$(D_k^\nu\setminus\{(\lambda_k,\infty)\})\subset R\times S_k^\nu.$$

The proof of this lemma is complete.

Remark 1. In view of Lemma 2 alternative (i) of Theorem 2 cannot hold.

Corollary 1. *If C_k^ν meets (λ,∞) for some $\lambda\in R,$ then $\lambda=\lambda_k$. If D_k^ν meets $(\lambda,\tilde{0})$ for some $\lambda\in R,$ then $\lambda=\lambda_k$.*

Thus we can prove the main result of this paper.

Theorem 3. *For each $k \in \mathrm{Z}$ and each $\nu \in \{+,-\}$ the following relation holds:*

$$C_k^\nu = D_k^\nu. \tag{9}$$

Proof. Let $\varepsilon_0 > 0$ be the fixed sufficiently small number. By conditions (4) and (5) there exit sufficiently small $\delta_0 > 0$ and sufficiently large $\Delta_0 > 0$ such that

$$\left|\frac{g(x,w)}{w}\right| < \varepsilon_0 \text{ for any } (x,w) \in [0,\pi]\times R^2, \; 0 < |w| < \delta_0 \text{ and } |w| > \Delta_0. \tag{10}$$

Since $g \in C([0,\pi]\times R^2)$ it follows that there exists $\kappa_0 > 0$ such that

$$\frac{|g(x,w)|}{|w|} < \kappa_0 \text{ for any } (x,w) \in [0,\pi]\times R^2, \; \delta_0 \le |w| \le \Delta_0. \tag{11}$$

We introduce the notation:

$$K = \max\{\varepsilon_0, \kappa_0\}.$$

Then by (10) and (11) we get

$$\frac{|g(x,w)|}{|w|} < K \text{ for any } (x,w) \in [0,\pi]\times R^2, \; |w| \ne 0. \tag{12}$$

Let the projection $P_{R\times\{\tilde{0}\}}(C_k^\nu)$ of C_k^ν onto $R\times\{\tilde{0}\}$ is unbounded. Then there exists

$$\{(\lambda_n, \tilde{w}_n)\}_{n=1}^{\infty} \subset C_k^\nu \setminus \{(\lambda_k, \tilde{0})\}$$

such that

$$\lim_{n\to\infty} \tilde{\lambda}_n = -\infty \quad \text{or} \quad \lim_{n\to\infty} \tilde{\lambda}_n = +\infty. \tag{13}$$

For each $n \in \mathrm{N}$ we define the vector-function

$$\Phi_n(x) = \begin{pmatrix} \phi_n(x) & \varphi_n(x) \\ \psi_n(x) & \chi_n(x) \end{pmatrix}$$

as follows:

$$\phi_n(x) = \frac{g_1(x, \tilde{u}_n(x), \tilde{\vartheta}_n(x))\tilde{u}_n(x)}{\tilde{u}_n^2(x) + \tilde{\vartheta}_n^2(x)}, \quad \varphi_n(x) = \frac{g_1(x, \tilde{u}_n(x), \tilde{\vartheta}_n(x))\tilde{\vartheta}_n(x)}{\tilde{u}_n^2(x) + \tilde{\vartheta}_n^2(x)}, \tag{14}$$

$$\psi_n(x) = \frac{g_2(x, \tilde{u}_n(x), \tilde{\vartheta}_n(x))\tilde{u}_n(x)}{\tilde{u}_n^2(x) + \tilde{\vartheta}_n^2(x)}, \quad \phi_n(x) = \frac{g_2(x, \tilde{u}_n(x), \tilde{\vartheta}_n(x))\tilde{u}_n(x)}{\tilde{u}_n^2(x) + \tilde{\vartheta}_n^2(x)}. \tag{15}$$

By (12) it follows from (14) that

$$|\phi_n(x)| = \frac{|g_1(x,\tilde{u}_n(x),\tilde{\vartheta}_n(x))||\tilde{u}_n(x)|}{\tilde{u}_n^2(x)+\tilde{\vartheta}_n^2(x)} \le \frac{|g(x,\tilde{w}_n(x))||\tilde{u}_n(x)|}{\tilde{u}_n^2(x)+\tilde{\vartheta}_n^2(x)} \le$$

$$\frac{K|\tilde{w}_n(x)||\tilde{u}_n(x)|}{|\tilde{w}(x)|^2} = \frac{K|\tilde{u}_n(x)|}{|\tilde{w}(x)|} \le \frac{K|\tilde{w}(x)|}{|\tilde{w}(x)|} = K, \quad x\in[0,\pi]. \tag{16}$$

In a similar way, by (14) and (15), we can show that

$$|\varphi_n(x)| \le K,\ |\psi_n(x)| \le K,\ |\chi_n(x)| \le K \quad \text{for} \quad x\in[0,\pi]. \tag{17}$$

By (14) and (15) it follows from (1)-(3) that $(\tilde{\lambda}_n,\tilde{w}_n)\in R\times S_k^\nu$, $n\in\mathrm{N}$, solves the following linear spectral problem

$$\begin{cases} \ell(w)(x) - \Phi_n(x)\,w(x) = \lambda w(x),\ x\in(0,\pi), \\ U(w) = \tilde{0}. \end{cases} \tag{18}$$

Then by [3, Theorem 2.4 and Remark 2.1] from (18) we obtain

$$\theta_n'(x) = \lambda + (p(x)+\phi_n(x))\cos^2\theta_n(x) + (r(x)+\chi_n(x))\sin^2\theta_n(x) +$$
$$\frac{1}{2}(\varphi_n(x)+\psi_n(x))\sin 2\theta_n(x), n\in\mathrm{N},\ x\in(0,\pi), \tag{19}$$

where

$$\theta_n(x) = \theta(\tilde{w}_n, x), x\in[0,\pi], n\in\mathrm{N}.$$

Inteqrating both sides of relation (19) from the range of 0 to π and taking into account (8) we get

$$-\beta + k\pi + \alpha = \tilde{\lambda}_n + \int_0^\pi (p(x)+\phi_n(x))\cos^2\theta_n(x)\,dx + \int_0^\pi (r(x)+\chi_n(x))\sin^2\theta_n(x)\,dx +$$
$$\frac{1}{2}\int_0^\pi (\varphi_n(x)+\psi_n(x))\sin 2\theta_n(x)dx, n\in\mathrm{N}. \tag{20}$$

Then by (16) and (17) from (20) we obtain

$$|\tilde{\lambda}_n| \le |k|\pi + |\alpha-\beta| + 5K\pi,\ n\in\mathrm{N},$$

which contradicts the relation (13), and consequently, the projection $P_{R\times\{\tilde{0}\}}(C_k^\nu)$ of C_k^ν onto $R\times\{\tilde{0}\}$ is bounded. Then, by Theorem 1 and Corollary 1, C_k^ν meets the point (λ_k,∞) with respect to the set $R\times S_k^\nu$. Next, in a similar way we can show that the projection $P_{R\times\{\tilde{0}\}}(D_k^\nu)$ of D_k^ν onto $R\times\{\tilde{0}\}$ is bounded, and consequently, by Lemma 2 and Corollary 1, the continuum D_k^ν meets the point $(\lambda_k,0)$ with respect to the set $R\times S_k^\nu$. Hence by these arguments for each $k\in Z$ and each $\nu\in\{+,-\}$ the continuum C_k^ν coincides with the continuum D_k^ν. The proof of this theorem is complete.

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