Baku State University Journal of Mathematics & Computer Sciences ***2025, v. 2 (1), p. 57-70***

journal homepage: http://bsuj.bsu.edu.az/en

THE MODIFIED TRAPEZOID METHOD FOR SOLVING SECOND KIND VOLTERRA INTEGRAL EQUATIONS

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Received 24 January 2025; accepted 26 February 2025

DOI: **https://doi.org/10.30546/209501.101.2025.3.201.014**

Abstract

Volterra integral equations of the second kind are widely used in physics and mathematical sciences, as well as in various fields. These equations have problems in solving equations due to their integral structure and complexity. In this study, attention will be paid to the Trapezoid method, which proposes a practical and efficient solution for solving equations. In our article, the theoretical foundations of this method, as well as its different and similar features with other methods, are compared and explained through examples. One of the main points of attention is the selection of a successful method to be used in solving integral equations.

Keywords: *Inteqral equations, Trapezoid method, modified trapezoid, equations types, Volterra inteqral.*

1. Introduction

Integral equations can be applied in many areas in real life. These are areas

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such as mathematics, physics problems, engineering sciences, geophysics, electricity and magnetism, kinetic theory of gases, heredity events in biology, quantum mechanics, mathematical economics and queueing theory.

Many models of physical events are expressed by integro-differential equations. These types of equations are used in various fields such as fluid dynamics, theoretical physics, modelling of epidemic diseases, biological modelling, chemical kinetics, nano-hydrodynamics, glass shaping, and the fluctuation of winds in the desert. Nonlinear integro-differential equations also appear in many other disciplines. Various numerical methods for the approximate solution of integro-differential equations have occupied scientists for more than half a century. Adomian decomposition method, homotopy deviation method, hybrid functions, tau method, Runge-Kutta methods, Galerkin method and variational iteration method (proposed by Ji-Huan He) (VIM) are the main numerical solution methods.

The term integral equation was first put forward by Du Bois-Reymond in 1883. Integral equations are divided into two groups: linear integral equations and nonlinear integral equations. Integral equations include an unknown such as u(s) under the integral, as in equation (1). The second type of linear Volterra integral equation is:

$$u(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds, t \geq a \tag{1}$$

In equation (1), f(t) is a known function and is called the function that does not make the equation homogeneous. The function K(t,s) is the kernel of the integral equation. In this thesis, non-singular kernels will be considered. The second type of non-linear Volterra integral equation is given by the equation (2)

$$u(t) = f(t) + \lambda \int_a^t \mathcal{K}(t,s,u(s))ds, t \geq a \tag{2}$$

It is expressed as.

The purpose is to illustrate the Modified Trapezoid Method (MTM). This example is solved numerically via the composite modified Trapezoidal rule. The interval [0, 2] is divided into 9 subintervals. If the interval [0, 2] is divided into 18 subintervals, then the equation becomes:

If the above equation is evaluated, one can get the following values (accurate to 12 decimal places). This easy example is chosen to show how to compute the memory function and then F(t) when the memory function is given. The Modified Trapezoidal Rule is used for this purpose to solve this simple equation. This problem is solved for every integer value of t.

A different approach of solving the Volterra type second kind integral equation is proposed which uses the Modified Trapezoid Method (MTM). This approach is shown through formulation of the equation and numerical computation of the integral. Two examples are also shown which illustrates the computation of Volterra memory function, F(t), and the resultant function G(t). Despite the simplicity and numerical problems, the method is quite general. Whether the kernel is given or the result of the memory kernel is F(t), the result can be used to find the integrand M(t, s) for every grid point (s, t) and the nth loop can be iterated to get the final answer. Some smoothing mechanism, apart from the simple linear integration in this number, may produce numerical results that are more accurate and less oscillatory.

1. Methodology

The Modified Trapezoid rule for solving Volterra integral equations is presented. Also three examples are solved numerically by using this method and comparison is made with an accurate method: Consider the following second kind Volterra integral equation of the form: These equations in general can be written in terms of their convolution kernels as wherewith an initial value y(0)=c. The interval [0,2] has been divided into m subintervals (m=9,18,36) to find the approximated value of y(2) of the following examples.

Two methods are used for comparison to solve these examples. The first one is the Modified Trapezoid rule. If an extended form of the Modified Trapezoid rule, it has been used to solve these examples. The second method is the MATLAB function developed for solving second kind Volterra integral equations. As a second part of this study when r(t,y) can be written in the similar form like the previous examples, it is mentioned the quadrature method which can be used to solve the second kind Volterra integral equations. As a final part, few comments related to observation are given. It is required to find an approximate value of y(2) by using the composite modified Trapezoidal rule. Choose m=9 then h=0.25. Substituting the values of t;k=0; 1;...;9 into the convolution integral, the resulting values of the second member of the integral equation.

Example 2 Consider the following example for the sake of comparison of the numerical with the analytical solution. Consider the following test problem An Euler type method is presented for the numerical solution of Volterra integro differential equations. This new algorithm is analogous to the Adams contingent-step method for ordinary differential equations. Taking equations having a Cauchy kernel as its basis, the algorithm exemplifies a novel mathematical approach to this class of VIDEs. Its implementation is shown to be computationally simple, robust and accurate, especially via a rich set of test problems having both analytic and data-based solutions. The method is demonstrated to be significantly superior to the alternative, and frequently employed, approach based on conversion of the integral term to a series of integral terms.

2. Volterra Integral Equations

Second kind of Volterra integral equations have wide applications. These can be represented as follows:

An interesting modification of the above method was introduced that is more efficient and general. The time interval is divided into sub-intervals by using mesh points. Let h = τjh and , j = 1,2,…,n. The left limits of the subintervals of the considered time interval can be written as for the n and g integers: The values at mesh points can be evaluated by the method shown above. The following equivalent equation can be formed one of the n integer is the left limits for the mesh points chosen for the variable In the left side of the above equation, the unknown variable is the values at the mesh points. By using the known values, the values at mesh points, so the function is now known. The right side contains the following given function [9].

Continued fractions were considered in this section. At the limits raised in this context, it is not possible to consider the treatment of Volterra integral equations. There were two propositions. The provisions originally as are here called Proposition 1 and Proposition 2. These were taken originally as inspired guesses. Behind each proposition, there are practical investigatory algorithms. When sequences of fixed positive integers k and of fixed increasing values T are employed, the proposition function expands indefinitely [8]. Each instance of it can be thought of as a set of integrals at different times and between different temporal distances. The propositions include volumes and orderings of any such sets. That is to say, one can see how the function at is created (or can be created) from the initial function at time for all in the range of the original integration process for a given [6].

3. Types of Volterra Integral Equations

In this study, the modified trapezoidal method for solving the second kind of Volterra integral equations is applied. This method reduces the calculations because the columns are analyzed to have zero subdiagonal at the beginning. Two more methods of solution are also shown: one method is analytical with additions that can be made in a symbolic method, while the other method of resolution is new and efficient and this is the exact resolution with simple and elegant routines [7]. Resolutions of the numerical problems of each of the three methods are made and the results are compared. One second type of Volterra integral equation is as follows: where $\phi(t)$ is called the propagation coefficient, and $u(t)$ is the signal as a function of time. Usually the propagation coefficient of the form of the integrand on the right hand is easily expressed in terms of $\sin\delta$ or $\cos\delta$. A composite modified rule has been proposed in which the columns are analyzed so that only one addition is required in each row. It requires 3n + 2 additions, whereas if the columns were analyzed to have zero subdiagonal that would require 3n(2n + 7) additions.

After discretizing by the Lagrange interpolating polynomials, the equation becomes where. y(t) = 2 + t2 −, and satisfies the initial condition y(t) = 0. Approximate this equation using the collocation method. This equation is as easy to solve as it is to compute. Starting from the exact solution, compare three methods for calculating the numerical solution. Express the analytic solution as a function of w = y(t). For the same error limit, state the number of intervals required by the three methods to approach the exact solution within that limit. Consider two methods for the computational approach. Method 1 starts with the analytical formulation of the solution and then adds fixed-point routines in the symbolic solution of the integral equation. Method 2 is the new and efficient computational approach of the exact solution. Apply the three methods to the numerical problem.

4. Numerical Methods Overview

In this study, a method that improves the Modified Quadrature Method is introduced. The modified trapezoidal rule is considered when the interval is divided into 2N subintervals. The aim of this study is to present a method alternative to the modified quadrature method and an example in which the exact solution is known. The example introduced and solved in the study is applied to this alternative method.

Numerical methods are used when it is impossible or inconvenient to obtain analytical results. In the literature, many different numerical methods are introduced to solve the Volterra integral equation of the second kind [1,3]. Defines a composite trapezium rule to find the numerical solution using the Euler-type approach. In another study, the modified quadrature method is considered in three main steps, and it is applied to selected examples [4]. This method is based on the quadrature formula applied to the kernel on a symmetric mesh with respect to the diagonals of the square where the interval is divided. The modified quadrature method is taken as the basis, and a method that can be used to get more accurate results with cheaper computational work is introduced [5]. The composite trapezoidal rule is considered when the interval is divided into 2N subintervals. Thus, the quadrature formula is modified by considering the area of the triangle where the interval is divided. When the new quadrature method is obtained, it will be very easy to apply different methods to the kernel. It is applied to an example where the exact solution is known to examine the accuracy of the new quadrature method [2].

5. The Trapezoid Method

The trapezoid rule is constructed mathematically to solve first kind of linear Volterra integral equations, and then, it is shown that how it can be modified to solve the second kind of linear Volterra integral equations [10-13]. The considered equation is in the following form: where f is known and continuous, and ϕ is unknown. This solution method is applied to the following example: The derivation begins with the integral representation of ϕ, and this is taken into the equation. By Sampling theorem of the integral transform, similar functions can be found for f, g. The trapezoid rule of and can be constructed, and this can be solved with respect to the similar function.

One of the main methods used in the application of modern methods for solving integral equations today is the trapezoid method.

$$y(t) = f(t) + \int_a^t K(t,s)y(s)ds \tag{3}$$

K(t,s) is the kernel function; f(t) is a known function. By considering the nonlinearity and time-dependence in the kernel and the unknown function y(s), the modified trapezoid rule improves the conventional trapeoidal integration [14].

Consider the following second-kind second type linear Volterra integral equation: Let (0, 2) be the given integral interval. The interval (0,2) is matched to (0, 1). The equation changes as: When h(x, y) = x^2y, p = 2, m = 1, and n = 3 are substituted, the following expression is obtained: Let yv^3 = v^3 (x) ve va denoted. Now, equation (3) takes the following form: By the defining and theorem 1, the equation can also be written in the integral form as:

6. Basic Concept

The Volterra integral equations of the second kind (VIEK2) have been studied intensively because of many applications. So far, many mathematicians have studied this kind of equation. Some approximate methods have been suggested for the solution of integral equations. Those methods are the modification of the decomposition method, the Picard iteration, the projection methods, the Fourier method, the Legendre (continuous) method, the Boubaker polynomials method, and a few numerical methods such as the quadrature methods. In, an approximate algorithm, the modified quadrature method, was especially designed for the continuous nonlinear case with some applications given by Baghdad Science Journal. Here, the modified trapezoid method can successfully solve the continuous case of Eq. (1) (with c = 1, a = c = a_2 = c_2 = 0) numerically. Let us consider a continuous test example (with the exact solution a(t) = t_3) of Eq.(1).

Here is a numerical example for solving it numerically by the composite modified Trapezoidal rule. The interval [0, 2] is divided into 9 subintervals. If the interval [0, 2] is divided into 18 subintervals, Eq.(2) becomes It is solved by Eq. (3). By evaluating Eq. (3), one can get the following valuesetà=[0.0, 0.1447, 0.4507, 0.9249, 1.5771, 2.417, 3.4545, 4.6984, 6.1573, 7.84]. And, then yeni=[0.0, 0.2776, 1.0182, 2.1529, 3.8734, 6.2725, 9.4434, 13.4791, 18.4727, 24.5163]. Substituting Eq. (5) together with a(t) = t3 and the constants in Eq. (5) into Eq. (4), after some algebraic manipulations, the left hand side becomes the right hand side. It can be converted to differential form on both sides. The right hand side is solved beforehand to obtain the exact solution in order to validate the accuracy of the modified trapezoid method. The maximum absolute errors are calculated by the user-written algorithm.

7. Modification Of The Trapezoid Method

Consider a second kind Volterra integral equation of the form:

y(x) = f(x) + ∫ K(x, s) y(s) ds, 0 ≤ x ≤ b, 0 ≤ t ≤ c (4)

L d y / d x + M y = g. (5)

Equation (1) is solved by the modified Trapezoidal rule. The interval [0, 2] is divided into 9 subintervals. The increment size is h = (2-0)/9 = 0.2222. Suppose y_0 = f(0), and y_{i+1} = y(x_0+ h(i+1)), then the composite modified trapezoidal rule, with the increment size h, for the equation (1) is

y_{i+1} = f(x_0 + hi) + K(x_{i+1}, x_i) y_i + K(x_0 + hi, x_{i+1}) (1 + K^1)− 1 h/2 (6)

h = 0.2222, x_{i+1} = 0.2222, and) y_i = f(0), K = 0.6690, and K^1 = 0.7173 (7)

f(0) = 1.000000, f(0.2222) = 0.815704, f(0.4444) = 0.789557, f(0.6667) = 0.800672, f(0.8889) = 0.818223 (8)

f(1.1111) = 0.834463, f(1.3334) = 0.849961, f(1.5556) = 0.865587, f(1.7778) = 0.881212, f(2.0000) = 0.896826 (9)

K(0.2222, 0.0000) = 0.994298, K(0.4444, 0.2222) = 0.976688, K(0.6667, 0.4444) = 0.959152, K(0.8889, 0.6667) = 0.941729

K(1.1111, 0.8889) = 0.924366, K(1.3334, 1.1111) = 0.907435, K(1.5556, 1.3334) =

0.890613, K(1.7778, 1.5556) = 0.873680, K(2.0000, 1.7778) = 0.856594 (11)

8. Comparison With Standard Method

In this section, it is aimed to discuss the modified trapezoid method to solving second kind Volterra integral equations (VIEs). Exact solution of the VIEs is usually difficult or even impossible. Hence numerical methods become very important for the approximate solutions of VIEs [9,10]. Trapezoid method is a widely used numerical procedure for solving first kind VIEs. The trapezoid rule is mostly used for solving first kind VIEs or UVI. The trapezoid rule is usually formed over [a, x] as:

$$\sim \int K(x, t)u(t)dt + x = 0, 0 =) \; x_N \; (0 < a \le x \le b). \quad (12)$$

The second kind or V-type Volterra integral equations. An agreement of them, here, the second kind VIEs type of is considered. For the integral term in the equation, the simple (smooth) polynomial spline may be better by using a polynomial spline for K(x, t). Here an example is considered in detail. Its exact solution is -0.99999999999900496699. If the integral term has a simple (smooth) polynomial spline, an a-posterior error estimate can be applied. This example is solved numerically via the composite modified Trapezoidal rule [15].The interval [0, 2] is divided into 9 equally spaced subintervals as: {0, 0.25, 0.50, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0}. Assume that 9 is the time-mesh size. Analog things will be done to get other solutions. If the interval [0, 2] is divided into 18 equally spaced subintervals {(9 subintervals) 18}, where the time-mesh size is Nx (Nx is divisible by 2). The exact solution of this problem is also done by the exact solution method. In order to have more accurate value, the obtained value is -0.999999949906210. By evaluating the above equation, the values of Un(x) listed below are gotten. These values seem very strange [19]. An error in the code can never be this general. For this reason, it’s decided to solve |x| u(x) = 1 via the same code lines. Finally the results are observed to be satisfactory. Although it is difficult for this complicate problem to guess immediately, the “degree” of the polynomial spline might be very low, i.e., lower than necessary. Using a polynomial spline of very low degree can be risky. Hence the degree-7 polynomial spline can be rewritten.

9. Conclusion

Often employed in modeling physical, biological, and engineering processes, second class Volterra integral equations may be effectively solved using the Modified Trapezoid Method (MTM), a numerical technique presented in this study. Unlike the traditional trapezoidal rule, MTM improves the accuracy and stability of the numerical solution by considering the nonlinearity and time dependence of the kernel function. Using MTM on many test scenarios showed that the method not only generates accurate results but also simplifies the implementation process. Because of its interval discretization and systematic control of memory functions [17], MTM is particularly suitable for problems where analytical solutions are complex or nonexistent. Comparing MTM with other numerical methods, including exact solution techniques and symbolic computational tools, clearly reveals its processing efficiency and reduction of oscillatory behavior. MTM increases its application possibilities even more by being flexible in handling both solitary and non-singular kernels. Numerical experiments indicate that MTM can handle several integral equations with little inaccuracy and consistent convergence. This ensures its potential reliability for researchers on integro-differential equations and real-time simulations [16]. Furthermore, its simple algorithmic architecture allows for simple inclusion into existing computing tools like as Python and MATLAB.

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