Baku State University Journal of Mathematics & Computer Sciences ***2024****, v 1 (2), p. 27-39*

journal homepage: http://bsuj.bsu.edu.az/en

SEMILINEAR HYPERBOLIC EQUATIONS WITH NONLINEAR ACOUSTIC CONDITION

Sevda E. Isayeva*

Baku State University

Received 25 January 2024; accepted 02 May 2024

Abstract

A mixed problem for nonlinear hyperbolic equations with nonlinear acoustic transmission condition is considered. The theorem on existence and uniqueness of solutions for this problem is proved by the semigroup method.

Keywords *Nonlinear hyperbolic equations, nonlinear transmission acoustic condition, contraction semigroup, weak solution, dissipative operator, Lumer-Phillips theorem.*

Mathematics Subject Classification *(2019): 35A01, 35A02*

1. Introduction

Let Ω be a bounded domain in $\Omega \subset R^3$ with smooth boundary Γ_1, $\Omega_2 \subset \Omega$ is a subdomain with smooth boundary Γ_2 and $\Omega_1 = \Omega \setminus (\Omega_2 \bigcup \Gamma_2)$ is the subdomain with boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$, $\Gamma_1 \bigcap \Gamma_2 = \emptyset$.

The nonlinear transmission acoustic problem considered here is

$$u_{tt} - \Delta u + \alpha_1 u_t + u + f_1(u) = 0 \quad \text{in} \quad \Omega_1 \times (0, \infty), \tag{1}$$

* *E-mail address:* isayevasevda@rambler.ru

$$\upsilon_{tt} - \Delta\upsilon + \alpha_2\upsilon_t + \upsilon + f_2(\upsilon) = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty) \,, \tag{2}$$

$$\delta_{tt} + \beta\delta_t + \delta = -u_t \quad \text{on} \quad \Gamma_2 \times (0,\infty) \,, \tag{3}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty) \,, \tag{4}$$

$$u = \upsilon \,,\ \delta_t = \frac{\partial u}{\partial \nu} - \frac{\partial \upsilon}{\partial \nu} + \rho(u_t) \quad \text{on} \ \Gamma_2 \times (0,\infty) \,, \tag{5}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega_1, \tag{6}$$

$$\upsilon(x,0) = \upsilon_0(x),\ \upsilon_t(x,0) = \upsilon_1(x), \quad x \in \Omega_2, \tag{7}$$

$$\delta(x,0) = \delta_0(x),\ \delta_t(x,0) = \frac{\partial u_0}{\partial \nu} - \frac{\partial \upsilon_0}{\partial \nu} + \rho(u_1) \equiv \delta_1(x), \quad x \in \Gamma_2, \tag{8}$$

where ν is the outward normal to the boundary Γ; $\alpha_i > 0 \ (i = 1,2)$ and $\beta > 0$ are constants; $f_i, \rho: R \to R \ (i = 1,2)$, $u_0, u_1: \bar{\Omega}_1 \to R$, $\upsilon_0, \upsilon_1: \bar{\Omega}_2 \to R$, $\delta_0: \Gamma_2 \to R$ are given functions.

The problems like (1)-(8), called transmission acoustic problems, are related to the problem of two wave equations which models the transverse acoustic vibrations of the membrane composed by two different materials Ω_1 and Ω_2.

Transmission problems were studied, for example, in [4-6, 13]. The acoustic boundary conditions were studied in [1-3], [7-8], [11-12].

The problems like (1)-(8) with linear acoustic conditions were studied in [23-25] in which some results on local existence, global existence, the exponential stability and blow up results were obtained.

In this paper we prove the theorem on existence and uniqueness of weak solutions for the problem (1)-(8) with nonlinear acoustic transmission condition.

2. Preliminary

The inner product and norm in $L^2(\Omega_i)$, $i = 1,2$ and $L^2(\Gamma_2)$ are denoted respectively, by

$$(u,\upsilon)_i = \int_{\Omega_i} u(x)\upsilon(x)dx, \ \|u\|_i = \left(\int_{\Omega_i} (u(x))^2 dx \right)^{1/2}, \ i = 1,2,$$

$$(\delta,\theta)_{\Gamma_2}=\int_{\Gamma_2}\delta(x)\theta(x)d\Gamma_2,\ \ \|\delta\|_{\Gamma_2}=\left(\int_{\Gamma_2}(\delta(x))^2\,d\Gamma_2\right)^{1/2}.$$

$H^1(\Omega_i)$, $i=1,2$ are the usual real Sobolev spaces of first order. We define a closed subspace of the space $H^1(\Omega_1)$ as

$$H^1_{\Gamma_1}(\Omega_1)=\{\,u\in H^1(\Omega_1):\gamma_0(u)=0\ \text{a.e. on }\Gamma_1\},$$

where $\gamma_0: H^1(\Omega_1)\to H^{1/2}(\Gamma)$ is the trace map of order zero and $H^{1/2}(\Gamma)$ is the Sobolev space of order $\frac{1}{2}$ defined over Γ, as introduced by Lions and Magenes [10]. Observe that the norm in $H^1_{\Gamma_1}(\Omega_1)$:

$$\|u\|_{H^1_{\Gamma_1}(\Omega_1)}=\left(\sum_{i=1}^{n}\int_{\Omega_1}\left(\frac{\partial u}{\partial x_i}\right)^2 dx\right)^{1/2}$$

and the norm of the real Sobolev space $H^1(\Omega_1)$ are equivalent, because the Poincaré's inequality holds in $H^1_{\Gamma_1}(\Omega_1)$. Thus, we consider $H^1_{\Gamma_1}(\Omega_1)$ with the above gradient norm.

The map $\gamma_1: H(\Delta,\Omega_1)\bigcup H(\Delta,\Omega_2)\to H^{-1/2}(\Gamma_2)$ is the Neumann trace map on $H(\Delta,\Omega_1)\bigcup H(\Delta,\Omega_2)$ and

$$H(\Delta,\Omega_i)=\{\,u\in H^1(\Omega_i):\Delta u\in L^2(\Omega_i)\,\},\ \ i=1,2$$

are equipped with the norms

$$\|u\|_{\Delta,\Omega_i}=\left(\|u\|^2_{H^1(\Omega_i)}+\|\Delta u\|_i^2\right)^{1/2},\ i=1,2.$$

Assume that

$$f_i\in C^1(R),\ |f_i'(s)|\le c_{1i}\left(1+s^2\right),\ c_{1i}\ge 0,\ i=1,2, \tag{9}$$

$$\liminf_{|s|\to\infty}\frac{f_i(s)}{s}>-1,\ i=1,2. \tag{10}$$

$$\rho\in C^1(R),\ |\rho(s)|\le c|s|,\ c\ge 0, \tag{11}$$

$\rho(s)$ is monotone increasing function on $(-\infty,+\infty)$ with

$$\rho(0)=0. \tag{12}$$

For some results instead of (9), we assume that the nonlinearities $f_i \in C^2(R)$ satisfy the conditions

$$\left| f_i''(s) \right| \le c_{2i}\left(1+|s|\right),\ c_{2i} \ge 0,\ i=1,2. \tag{13}$$

We will form the initial-boundary problem (1)-(8) in the phase space

$$V=\Big\{ w=\left(w_1,w_2,w_3,w_4,w_5,w_6\right): w_1 \in H^1_{\Gamma_1}\left(\Omega_1\right), w_2 \in L^2\left(\Omega_1\right),$$

$$w_3 \in H^1\left(\Omega_2\right), w_4 \in L^2\left(\Omega_2\right), w_5 \in L^2\left(\Gamma_2\right), w_6 \in L^2\left(\Gamma_2\right), w_1\big|_{\Gamma_2} = w_3\big|_{\Gamma_2} \Big\},$$

which is Hilbert space with the norm

$$\|w\|_V^2 = \|w_1\|^2_{H^1_{\Gamma_1}(\Omega_1)} + \|w_2\|^2_{L^2(\Omega_1)} + \|w_3\|^2_{H^1(\Omega_2)} + \|w_4\|^2_{L^2(\Omega_2)} + \|w_5\|^2_{L^2(\Gamma_2)} + \|w_6\|^2_{L^2(\Gamma_2)}$$

for $\forall w=\left(w_1,w_2,w_3,w_4,w_5,w_6\right)\in V$. We introduce the functional

$$E_w(t)=\frac{1}{2}\|w\|_V^2 + \int_{\Omega_1} F_1\left(w_1\right)dx \ + \int_{\Omega_2} F_2\left(w_3\right)dx,$$

where $F_i(s)=\int_0^s f_i(s)\,ds$, $i=1,2$, which formally satisfies the equality

$$\frac{dE_w}{dt} = -\alpha_1\|u_t\|_1^2 - \alpha_2\|\upsilon_t\|_2^2 - \beta\|\delta_t\|^2_{\Gamma_2} \tag{14}$$

for the solution $\left(u,u_t,\upsilon,\upsilon_t,\delta,\delta_t\right)$ of the problem (1)-(8).

We introduce the linear unbounded operator $A: D(A)\subset V \to V$,

$$Aw=\left(w_2, \Delta w_1 - w_1 - \alpha_1 w_2, w_4, \Delta w_3 - w_3 - \alpha_2 w_4, w_6, -w_2 - w_5 - \beta w_6\right),$$

$$D(A)=\Big\{ w=\left(w_1,w_2,w_3,w_4,w_5,w_6\right)\in V : \Delta w_1 \in L^2\left(\Omega_1\right),$$

$$w_2 \in H^1\left(\Omega_1\right), \Delta w_3 \in L^2\left(\Omega_2\right), w_4 \in H^1\left(\Omega_2\right),$$

$$w_2\big|_{\Gamma_2} = w_4\big|_{\Gamma_2},\ \ w_6 = w_{1\nu}\big|_{\Gamma_2} - w_{3\nu}\Big|_{\Gamma_2} + \rho\left(w_2\right)\big|_{\Gamma_2} \Big\}.$$

The condition $w_6 = w_{1\nu}\big|_{\Gamma_2} - w_{3\nu}\Big|_{\Gamma_2} + \rho\left(w_2\right)\big|_{\Gamma_2}$ interpreted in a weak sense as

$$\int_{\Omega_1}\left(\Delta w_1 \phi + \nabla w_1 \nabla \phi\right)dx + \int_{\Omega_2}\left(\Delta w_3 \psi + \nabla w_3 \nabla \psi\right)dx +$$

$$+\int_{\Gamma_2} \rho(w_2)\,\phi\, d\Gamma_2 = \int_{\Gamma_2} w_6\,\phi\, d\Gamma_2 \tag{15}$$

for $\forall \phi \in H^1(\Omega_1)$, $\forall \psi \in H^1(\Omega_2)$ such that $\phi|_{\Gamma_2} = \psi|_{\Gamma_2}$. We introduce the nonlinear function $\Phi : V \to V$ as

$$\Phi(w) = (0, -f_1(w_1), 0, -f_2(w_3), 0, 0) \text{ for } \forall w \in V.$$

Then the problem (1)-(8) has the following form

$$\begin{cases} w_t = Aw + \Phi(w), \\ w(0) = w_0, \end{cases} \tag{16}$$

where $w = (u, u_t, \upsilon, \upsilon_t, \delta, \delta_t)$ and $w_0 = (u_0, u_1, \upsilon_0, \upsilon_1, \delta_0, \delta_1) \in V$.

In order to consider strong solutions, we introduce the phase space

$$V_1 = \Big\{ w = (w_1, w_2, w_3, w_4, w_5, w_6) \in \left(H^1_{\Gamma_1}(\Omega_1) \cap H^2(\Omega_1)\right) \times H^1(\Omega_1) \times$$
$$\times H^2(\Omega_2) \times H^1(\Omega_2) \times H^{1/2}(\Gamma_2) \times H^{1/2}(\Gamma_2) :\ w_1|_{\Gamma_2} = w_3|_{\Gamma_2},\, w_2\Big|_{\Gamma_2} = w_4|_{\Gamma_2},$$
$$w_6 = w_{1\nu}|_{\Gamma_2} - w_{3\nu}\Big|_{\Gamma_2} + \rho(w_2)|_{\Gamma_2} \Big\}$$

which is Hilbert space with respect to the norm

$$\|w\|^2_{V_1} = \|w_1\|^2_{H^1_{\Gamma_1}(\Omega_1) \cap H^2(\Omega_1)} + \|w_2\|^2_{H^1(\Omega_1)} + \|w_3\|^2_{H^2(\Omega_2)} + \|w_4\|^2_{H^1(\Omega_2)} +$$
$$+ \|w_5\|^2_{H^{1/2}(\Gamma_2)} + \|w_6\|^2_{H^{1/2}(\Gamma_2)}.$$

By the smoothness of the boundary Γ, it is easy to see that

$$V_1 = D(A) \cap Z,$$

where Z is a subspace of the space V:

$$Z = \left\{ z = (w_1, w_2, w_3, w_4, w_5, w_6) \in V : w_5, w_6 \in H^{\frac{1}{2}}(\Gamma_2) \right\}.$$

Definition. Assume that $w_0 = (u_0, u_1, \upsilon_0, \upsilon_1, \delta_0, \delta_1) \in V$. The function $w \in C^0([0, \infty); V)$ is called a weak solution of the problem (16), if it satisfies the equality

$$w(t) = e^{At} w_0 + \int_0^t e^{A(t-s)} \Phi(w(s))\, ds$$

for $\forall t \geq 0$.

Let A^* be the adjoint of the operator A. It can easily be proved (see [16]) that the function $w \in C^0\left(\left[0,\infty\right);V\right)$ is a weak solution of the problem (16) only if for each $z \in D\left(A^*\right)$ the function $\left(w(\cdot),z\right)_V$ is absolutely continuous on $\left[0,T\right]$ for each $T > 0$ and satisfies the relation

$$\frac{d}{dt}\left(w(t),z\right)_V = \left(w(t),A^*z\right)_V + \left(\Phi\left(w(t)\right),z\right)_V$$

for almost all $t \in \left[0,\infty\right)$ and the initial condition $w(0) = w_0$.

3. Basic results

The existence and uniqueness of solutions, as well as the regularity of solutions of the problem (16) are established in the next theorem.

Theorem 1. Let the conditions (9)-(12) be satisfied and let $w_0 \in V$. Then there exists a unique weak solution $w \in C^0\left(\left[0,\infty\right);V\right)$ to the problem (16). Moreover, if $\bar{w}_1$, $\bar{w}_2$ are solutions on $\left[0,\infty\right)$ to the problem (16), corresponding to two initial data $\bar{w}_{10}$ and $\bar{w}_{20}$ with $\left\|\bar{w}_{10}\right\|_V \leq r$ and $\left\|\bar{w}_{20}\right\|_V \leq r$ ($r > 0$), then there exists a positive number θ depending on r such that for all $t \geq 0$

$$\left\|\bar{w}_2(t) - \bar{w}_1(t)\right\|_V \leq e^{\theta t}\left\|\bar{w}_{20} - \bar{w}_{10}\right\|_V . \tag{17}$$

If, in addition, we assume that the functions f_i $\left(i = 1,2\right)$ satisfy conditions (13) and that $w_0 \in V_1$, then the corresponding weak solution will possess the regularity property

$$w \in C^1\left(\left[0,\infty\right);V\right) \cap C^0\left(\left[0,\infty\right);V_1\right) \tag{18}$$

and is called a strong solution.

Theorem 1 and the fact that the system (14) is autonomous readily imply the following assertion.

Corollary. Under conditions (9)-(12), system (16) generates strongly continuous semigroup $S(t)$ in the phase space V. This semigroup is defined by the formula

$$S(t)\left(u_0,u_1,\upsilon_0,\upsilon_1,\delta_0,\delta_1\right) := \left(u,u_t,\upsilon,\upsilon_t,\delta,\delta_t\right)$$

where $\left(u, u_t, \upsilon, \upsilon_t, \delta, \delta_t\right) \in C^0\left(\left[0, \infty\right); V\right)$ is the weak solution of the problem (16) corresponding to the initial data $\left(u_0, u_1, \upsilon_0, \upsilon_1, \delta_0, \delta_1\right) \in V$.

Proof of Theorem 1. The fact that the set $D(A)$ is dense in V can be proved by analogy with [2, theorem 2.1]. The methods for proving the theorem in the above indicated paper can be used to prove the fact that the operator A is closed and dissipative, i.e., $(Aw, w) \le 0$ for each $w \in D(A)$. Indeed, since

$$\left(w_2, w_1\right)_{H^1_{\Gamma_1}(\Omega_1)} = \left(\nabla w_2, \nabla w_1\right)_1 + \left(w_2, w_1\right)_1,$$

$$\left(w_4, w_3\right)_{H^1(\Omega_2)} = \left(\nabla w_4, \nabla w_3\right)_2 + \left(w_4, w_3\right)_2 - \rho\left(w_2\right)$$

and using (15) for each $w \in D(A)$ we have the relation

$$\left(\Delta w_1, w_2\right)_1 + \left(\nabla w_1, \nabla w_2\right)_1 + \left(\Delta w_3, w_4\right)_2 + \left(\nabla w_3, \nabla w_4\right)_2 = \left(w_6 - \rho\left(w_2\right), w_2\right)_{\Gamma_2},$$

we see that the following relation holds for $\forall w \in D(A)$:

$$\begin{aligned}
\left(Aw, w\right) &= \left(w_2, w_1\right)_{H^1_{\Gamma_1}(\Omega_1)} + \left(\Delta w_1 - w_1 - \alpha_1 w_2, w_2\right)_1 + \left(w_4, w_3\right)_{H^1(\Omega_2)} + \\
&+ \left(\Delta w_3 - w_3 - \alpha_2 w_4, w_4\right)_2 + \left(w_6, w_5\right)_{\Gamma_2} - \left(w_2 + w_5 + \beta w_6, w_6\right)_{\Gamma_2} = \\
&= \left(\nabla w_2, \nabla w_1\right)_1 + \left(\Delta w_1, w_2\right)_1 - \alpha_1\left(w_2, w_2\right)_1 + \left(\nabla w_3, \nabla w_4\right)_2 + \\
&+ \left(\Delta w_3, w_4\right)_2 - \alpha_2\left(w_4, w_4\right)_2 - \left(w_2, w_6\right)_{\Gamma_2} - \beta\left(w_6, w_6\right)_{\Gamma_2} = \\
&= \left(w_6 - \rho\left(w_2\right), w_2\right)_{\Gamma_2} - \alpha_1\left(w_2, w_2\right)_1 - \alpha_2\left(w_4, w_4\right)_2 - \left(w_2, w_6\right)_{\Gamma_2} - \\
&- \beta\left(w_6, w_6\right)_{\Gamma_2} = \left(-\rho\left(w_2\right), w_2\right)_{\Gamma_2} - \alpha_1\left(w_2, w_2\right)_1 - \alpha_2\left(w_4, w_4\right)_2 - \\
&- \beta\left(w_6, w_6\right)_{\Gamma_2} = -\left(\rho\left(w_2\right), w_2\right)_{\Gamma_2} - \alpha_1\left\|w_2\right\|_1^2 - \alpha_2\left\|w_4\right\|_2^2 - \beta\left\|w_6\right\|_{\Gamma_2}^2 \le 0.
\end{aligned} \tag{19}$$

If we prove that the range of the operator $\lambda I - A$ is the entire space V, i.e., that $R(\lambda I - A) = V$, then according to the Lumer-Phillips theorem, this should imply that A is the generator of a contraction semigroup in V.

To this end, we write the equation $(\lambda I - A)w = g$ in an explicit form as

$$\lambda w_1 - w_2 = g_1 \text{ in } L^2(\Omega_1),$$

$$\lambda w_2 - \Delta w_1 + w_1 + \alpha_1 w_2 = g_2 \text{ in } L^2(\Omega_1),$$

$$\lambda w_3 - w_4 = g_3 \text{ in } L^2(\Omega_2),$$

$$\lambda w_4 - \Delta w_3 + w_3 + \alpha_2 w_4 = g_4 \text{ in } L^2(\Omega_2),$$
$$\lambda w_5 - w_6 = g_5 \text{ in } L^2(\Gamma_2),$$
$$\lambda w_6 + w_2\big|_{\Gamma_2} + w_5 + \beta w_6 = g_6 \text{ in } L^2(\Omega_2),$$

or if $\lambda \neq 0$ as

$$w_2 = \lambda w_1 - g_1 \text{ in } L^2(\Omega_1),$$
$$\left(\lambda^2 + \alpha_1\lambda + 1\right) w_1 - \Delta w_1 = g_2 + \lambda g_1 + \alpha_1 g_1 \text{ in } L^2(\Omega_1),$$
$$w_4 = \lambda w_3 - g_3 \text{ in } L^2(\Omega_2),$$
$$\left(\lambda^2 + \alpha_2\lambda + 1\right) w_3 - \Delta w_3 = g_4 + \lambda g_3 + \alpha_2 g_3 \text{ in } L^2(\Omega_2),$$
$$w_5 = \frac{1}{\lambda}\left(g_5 + w_6\right) \text{ in } L^2(\Gamma_2),$$
$$\left(\lambda + \frac{1}{\lambda} + \beta\right) w_6 + \lambda w_1\big|_{\Gamma_2} = g_1 - \frac{1}{\lambda} g_5 + g_6 \text{ in } L^2(\Omega_2).$$

Hence, considering the relation $w_6 = w_{1\nu}\big|_{\Gamma_2} - w_{3\nu}\Big|_{\Gamma_2} + \rho\left(w_2\right)\big|_{\Gamma_2}$, we conclude that the equation $(\lambda I - A)w = g$ is equivalent to the boundary value problem

$$\left(\lambda^2 + \alpha_1\lambda + 1\right) w_1 - \Delta w_1 = g_2 + \lambda g_1 + \alpha_1 g_1 \text{ in } L^2(\Omega_1), \tag{20}$$
$$\left(\lambda^2 + \alpha_2\lambda + 1\right) w_3 - \Delta w_3 = g_4 + \lambda g_3 + \alpha_2 g_3 \text{ in } L^2(\Omega_2), \tag{21}$$
$$\begin{aligned} w_{1\nu} - w_{3\nu} + \rho\left(w_2\right) + \left(\lambda^2 / \left(\lambda^2 + \beta\lambda + 1\right)\right) w_1 = \\ = \left(\lambda g_1 - g_5 + \lambda g_6\right) / \left(\lambda^2 + \beta\lambda + 1\right) \end{aligned} \tag{22}$$

in $L^2(\Gamma_2)$ for $\lambda \neq 0$, $w \in D(A)$, where

$$\begin{aligned} w_2 = \lambda w_1 - g_1,\ w_4 = \lambda w_3 - g_3,\ w_5 = \left(g_5 + w_6\right)/\lambda, \\ w_6 = w_{1\nu} - w_{3\nu} + \rho\left(w_2\right). \end{aligned} \tag{23}$$

In other words, the equation $(\lambda I - A)w = g$ is solvable, because there exist w_1 and w_3, from $H^1_{\Gamma_1}(\Omega_1)$ and $H^1(\Omega_2)$, respectively, such that the equations (20)-(22) are satisfied. The normal derivative is interpreted by analogy with (15). Consequently, the problem (20)-(22) is equivalent to the relation

$$\left(\left(\lambda^2+\alpha_1\lambda+1\right)w_1,\psi_1\right)_1+\left(\nabla w_1,\nabla\psi_1\right)_1+\left(\left(\lambda^2+\alpha_2\lambda+1\right)w_3,\psi_2\right)_2+$$
$$+\left(\nabla w_3,\nabla\psi_2\right)_2+\left(\rho\left(w_2\right)\big|_{\Gamma_2},\psi_1\right)_{\Gamma_2}+\left(\lambda^2/\left(\lambda^2+\beta\lambda+1\right)\right)\left(w_1,\psi_1\right)_{\Gamma_2}= \quad (24)$$
$$=\left(h_1,\psi_1\right)_1+\left(h_2,\psi_2\right)_2+\left(h_3,\psi_1\right)_{\Gamma_2}$$

for all $\psi_1\in H^1\left(\Omega_1\right)$, $\psi_2\in H^1\left(\Omega_2\right)$ such that $\psi_1=\psi_2\big|_{\Gamma_2}$, where

$$h_1=g_2+\lambda g_1+\alpha_1 g_1\in L^2\left(\Omega_1\right),$$
$$h_2=g_4+\lambda g_3+\alpha_2 g_3\in L^2\left(\Omega_2\right),$$
$$h_3=\left(\lambda g_1-g_5+\lambda g_6\right)/\left(\lambda^2+\beta\lambda+1\right)\in L^2\left(\Gamma_2\right).$$

If $\lambda>0$, the left-hand side of the relation (24) defines an inner product on $H^1_{\Gamma_1}\left(\Omega_1\right)\times H^1\left(\Omega_2\right)$ equivalent the ordinary one. Since the right-hand side is a continuous linear functional of $\left(\psi_1,\psi_2\right)$, it follows by the Riesz theorem that there exists a unique solution $\left(w_1,w_3\right)$ to the equation (24). This being done, the other components of the solution w, i.e., w_2,w_4,w_5,w_6 can be determined from the relations (23).

According to the condition (9), the function $\Phi:V\to V$ is locally Lipschitz continuous; therefore for each $w_0\in V$ there exists a $t_{\max}=t_{\max}\left(w_0\right)\in\left(0,+\infty\right]$ such that the problem (16) has a weak solution $w\in C^0\left(\left[0,t_{\max}\right);V\right)$. Let's prove that $t_{\max}=\infty$.

If $w=\left(u,u_t,\upsilon,\upsilon_t,\delta,\delta_t\right)\in V$ is a solution to the problem (16), as in the case of (19), we can obtain that

$$\frac{dE_w}{dt}=-\alpha_1\left\|u_t\right\|_1^2-\alpha_2\left\|\upsilon_t\right\|_2^2-\beta\left\|\delta_t\right\|_{\Gamma_2}^2-\left(\rho\left(u_t\right),u_t\right)_{\Gamma_2}; \quad (25)$$

integrating (25) from 0 to t, we have

$$\|w\|_V^2+2\int_{\Omega_1}F_1(u)\,dx+2\int_{\Omega_2}F_2(\upsilon)\,dx+$$
$$+\int_0^t\left(\alpha_1\|u_t\|_1^2+\alpha_2\|\upsilon_t\|_2^2+\beta\|\delta_t\|_{\Gamma_2}^2+\left(\rho(u_t),u_t\right)_{\Gamma_2}\right)d\tau= \tag{26}$$
$$=\|w_0\|_V^2+2\int_{\Omega_1}F_1(u_0)\,dx+2\int_{\Omega_2}F_2(\upsilon_0)\,dx$$

for all $t\in[0,t_{\max})$.

According to (10), there exist $\mu_i\in(0,1]$ and $k_{f_i}>0$ $(i=1,2)$ such that

$$2\int_{\Omega_1}F_1(u)\,dx\ge-(1-\mu_1)\|u\|_1^2-k_{f_1},$$
$$2\int_{\Omega_2}F_2(\upsilon)\,dx\ge-(1-\mu_2)\|\upsilon\|_2^2-k_{f_2}. \tag{27}$$

Using (27) in (26) and according to (9), it is easy to get that

$$\|w\|_V^2\le C\left(\|w_0\|_V\right)$$

for all $t\in[0,t_{\max})$ and it means that $t_{\max}=\infty$.

Now let's prove the estimate for continuous independence, i. e., (17). Setting

$$\tilde{w}=\bar{w}_2-\bar{w}_1=\left(u_2-u_1,u_{2t}-u_{1t},\upsilon_2-\upsilon_1,\upsilon_{2t}-\upsilon_{1t},\delta_2-\delta_1,\delta_{2t}-\delta_{1t}\right)=$$
$$=\left(\tilde{u},\tilde{u}_t,\tilde{\upsilon},\tilde{\upsilon}_t,\tilde{\delta},\tilde{\delta}_t\right)$$

and $\tilde{w}_0=\bar{w}_{20}-\bar{w}_{10}$, it is easy to get that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}\|_V^2+\alpha_1\|\tilde{u}_t\|_1^2+\alpha_2\|\tilde{\upsilon}_t\|_2^2+\beta\left\|\tilde{\delta}_t\right\|_{\Gamma_2}^2=$$
$$=\left(f_1(u_1)-f_1(u_2),\tilde{u}_t\right)_1+\left(f_2(\upsilon_1)-f_2(\upsilon_2),\tilde{\upsilon}_t\right)_2. \tag{28}$$

According to the condition (9), we obtain

$$\left|\left(f_1(u_1)-f_1(u_2),\tilde{u}_t\right)_1\right|\le C(R)\|\tilde{u}\|_1\|\tilde{u}_t\|_1,$$
$$\left|\left(f_2(\upsilon_1)-f_2(\upsilon_2),\tilde{\upsilon}_t\right)_2\right|\le C(R)\|\tilde{\upsilon}\|_2\|\tilde{\upsilon}_t\|_2, \tag{29}$$

where R is a positive number such that $\|\bar{w}_i(0)\|_V\le R$ for $i=1,2$. Then using Gromwell's lemma from (28) and (29) we obtain the validity of (17).

According to the assumptions (13) on f_i, the contraction $\Phi: D(A) \to D(A)$ is locally Lipschitz continuous. Then if $w_0 \in V_1$ (consequently, $w_0 \in D(A)$), then the corresponding weak solution w satisfies the relation

$$w \in C^1\left(\left[0,\infty\right);V\right) \cap C^0\left(\left[0,\infty\right);D\left(A\right)\right) \quad (30)$$

(see [21], Theorem 2.5.6).

Now we consider the following initial problem in $X = H^{1/2}\left(\Gamma_2\right) \times H^{1/2}\left(\Gamma_2\right)$:

$$\begin{cases} z_t = Kz + y, \\ z(0) = \begin{pmatrix} \delta_0 \\ \delta_1 \end{pmatrix}, \end{cases} \quad (31)$$

$$z(t) = \begin{pmatrix} z_5(t) \\ z_6(t) \end{pmatrix}, \ K = \begin{pmatrix} 0 & I \\ -I & -\beta I \end{pmatrix}, \ y(t) = \begin{pmatrix} 0 \\ -u_t(t) \end{pmatrix},$$

where u_t is the second component of the solution w. Then, according to the inclusion (30) we conclude that $y \in C^0\left(\left[0,\infty\right);X\right)$. Since $K \in L(X)$ (consequently, B is a generating operator of a uniformly continuous semi group in X) and $z(0) \in X$, we obtain that the problem (31) has a unique solution $z \in C^1\left(\left[0,\infty\right);X\right)$. Comparing the problem (31) with the problem (16) (the two equations in (31) are equivalent to the last two equations in (16)), by virtue of the uniqueness we conclude that

$$z_5(t) = \delta(t) \ \text{ and } \ z_6(t) = \delta_t(t)$$

for all $t \geq 0$. These identities, together with (30), yield the inclusion (18).

Theorem 1 is proved.

References

[1] Beale JT, Rosencrans SI. Acoustic boundary conditions. Bull. Amer. Math. Soc. **1974**, v. 80 (6), p. 1276-1278.

[2] Beale JT. Spectral properties of an acoustic boundary condition. Indiana Univ. Math. J. **1976,** v. 25 (9), p. 895-917.

[3] Beale JT. Acoustic scattering from locally reacting surfaces. Indiana Univ. Math. J. **1977**, v. 26, p. 199-222.

[4] Dautray R, Lions JL. Analyze et Calcul Numerique pour les Sciences et les Techniques. **1984**, v. 1, Masson, Paris.

[5] Muñoz Rivera JE, Portillo Oquendo H. The transmission problem of viscoelastic waves. Acta Applicandae Mathematicae. **2000**, v. 60, p. 1-21.

[6] Bae JJ. Nonlinear transmission problem for wave equation with boundary condition of memory type. Acta, Appl. Math. **2010**, v. 110 (2), p. 907-919.

[7] Frota CL, Vicente A. A hyperbolic system of Klein-Gordon type with acoustic boundary conditions. Int. J.Pure Appl. Math. **2008**, v. 47 (2), p. 185-198.

[8] Mugnolo D. Abstract wave equations with acoustic boundary conditions. Math. Nachr. **2006**, v. 279 (3), p. 299-318.

[9] Morse PM, Ingard KU. Theoretical Acoustic. MaGraw-Hill. **1968.**

[10] Lions JL, Magenes E. Nonhomogeneous boundary value problems and applications. I, Springer Verlag, Berlin, **1972**.

[11] Vicente A. Wave equation with acoustic/memory boundary conditions. Bol. Soc. Parana. Mat. **2009**, v. 27 (1), p. 29-39.

[12] Boukhatem Y., Benabderrahmane B. Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. Nonlinear Analysis. **2014**, v. 97, p. 191-209.

[13] Aliev AB, Mammadhasanov EH. Well-posedness of initial boundary value problem on longitudind impact on a composite linear viscoelastic bar. Mathematical Methods in the Applied Sciences. **2017**, v. 40 (14), p. 5380-5390.

[14] Ball JM. Contiuity properties and global attractors of generalized semiflows and the Navier-Stokes equations. Nonlinear Science. **1977**, v. 7 (5), p. 475-502, Corrected version appears in the book Mechanics: From Theory to Computation, Springer-Verlag, New-York. **2000**, p. 447-474.

[15] Ball JM. Global attractors for damped semilinear wave equations. Discrete Contin. Dyn. Syst. **2004**, v. 10 (2), p. 31-52.

[16] Ball JM. Strongly continuous semigroups, weak solutions, and the variation of constants formula. Proc. Amer. Math. **1977**, v. 63 (2), p. 370-373.

[17] Frigeri S. Attractors for semilinear damped wave equations with an acoustic boundary condition. J. Evol. Equ. **2010**, v. 10 (1), p. 29-58.

[18] Belleri V., Pata V. Attractors for semilinear strongly damped wave equation on R^n. Discrete Contin. Dyn. Syst. **2001**, v. 7, p. 719-735.

[19] Goubet O., Moise I. Attractor for dissipative Zakharov system. Nonlinear Anal. **1998**, v. 31, p. 823-847.

[20] Moise I., Rosa R., Wang X. Attractors for non- compact semigroups via energy equation. Nonlinearity. **1998**, v. 11, p. 1369-1393.

[21] Zheng S. Nonlinear Evolution Equations. Pitman Monographs and Surveys in Pure and Applied Mathematics. CHAPMAN and HALL/CRC Boca Raton Florida. **2004**, v. 133.

[22] Gal CG, Goldstein GR, Goldstein JA. Oscillatory boundary conditions for acoustic wave equations. J. Evol. Equ. **2003**, v. 3 (4), p. 623-635.

[23] Aliev AB, Isayeva SE. Exponential stability of the nonlinear transmission acoustic problem. Mathematical Methods in the Applied Sciences. **2018**, v. 41 (16), p. 7055-7073.

[24] Aliev AB, Isayeva SE. Existence and nonexistence of global solutions for nonlinear transmission acoustic problem. Turkish Journal of Mathematics. **2018**, v. 42, p. 3211-3231.

[25] Aliev AB, Isayeva SE. Attractors for semilinear wave equations with acoustic transmission conditions. Differential equations. **2020**, v. 56 (4), p. 447-461.