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INVERSE BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract

This paper is devoted to study the inverse boundary value problem for a third-order partial differential equation with nonlocal boundary conditions, including integral conditions. Using variable separation method and analytical methods is proved the existence and uniqueness of the classical solution of the considered problem .

Keywords: third-order partial differential equation, inverse boundary value problem, nonlocal boundary conditions, contraction mapping principle

Mathematics Subject Classification *(2020): 34B09, 34B27, 34L15, 35G31, 35J40, 35R30*

1. Introduction

Let $D_T = \{(x,t): 0 \le x \le 1, 0 \le t \le T\}$, where T is a some positive constant. For the equation

$$u_{ttt}(x,t) + u_{xx}(x,t) = a(t)u(x,t) + b(t)g(x,t) + f(x,t), \ \ (x,t) \in D, \tag{1}$$

consider the inverse problem with boundary conditions

$$u(x,0) = \varphi_0(x) + \int_0^T p_0(t)u(x,t)dt, \ \ u_t(x,T) = \varphi_1(x) + \int_0^T p_1(t)u(x,t)dt,$$

$$u_{tt}(x,0) = \varphi_2(x) + \int_0^T p_0(t)u(x,t)dt, \ \ 0 \le x \le 1, \tag{2}$$

Neumann condition

$$u_x(0,t) = 0, \ \ 0 \le t \le T, \tag{3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)\,dx, \ 0 \le t \le T, \tag{4}$$

and with additional conditions

$$u(0,t) = h_1(t), \ \ 0 \le t \le T, \tag{5}$$

$$u(1,t) = h_2(t), \ \ 0 \le t \le T, \tag{6}$$

where $f(x,t)$, $\varphi_i(x)$, $p_i(t), i = 0,1,2,$ $h_i(t),$ $i = 1,2,$ are given functions and $u(x,t),$ $a(t)$ and $b(t)$ are the desired functions.

Condition (4) is a non-local integral condition of the first kind, that is, it does not contain the values of the desired solution at the boundary points. Note that problems of type (1)-(6) arise when studying various issues in natural science, namely, when studying fluid filtration in porous media [2, 15], heat transfer in a heterogeneous medium [7, 16], moisture transfer in soil [8, 11], propagation of acoustic waves [13], questions of mathematical biology [12] and others.

Direct and inverse problems for partial differential equations with nonclassical boundary conditions were studied in [1, 3-7, 9, 11, 15, 16] using various methods. In this paper we use the Fourier method, analytical methods and methods of functional analysis, we study the inverse problem for equation (1) with boundary conditions (2), Neumann condition (3), nonlocal integral condition (4) and additional conditions (5), (6).

We introduce the notation:

$$C^{2,3}(D_T) = \{u(x,t) : u(x,t) \in C^2(D_T), u_{ttt}(x,t) \in C(D_T)\}.$$

By the classical solution of the inverse boundary value problem (1)-(6) we mean a triple $\{u(x,t),\, a(t), b(t)\}$ of functions $u(x,t) \in C^{2,3}(D_T),\ \ a(t) \in C[0,T],$ $b(t) \in C[0,T],$ satisfying equation (1) and conditions (2)-(6) in the usual sense.

2. Reduction of problem (1)-(6) to an auxiliary boundary value problem

To study problem (1)-(5), we first consider the following problem:

$$y'''(t) = a(t)y(t),\ 0 \le t \le T, \tag{7}$$

$$y(0) = \int_0^T p_0(t)y(t)dt,\ \ \ y'(T) = \int_0^T p_1(t)y(t)dt,\ y''(0) = \int_0^T p_2(t)y(t)dt, \tag{8}$$

where $a(t) \in C[0,T]$, $p_i(t),\ i = 0,1,2,$ are given functions and $y = y(t)$ is the desired function, where by the solution of problem (7), (8) we mean a function that is continuous on together with all its derivatives included in equation (7), and satisfies conditions (7), (8) in the usual sense.

Lemma 1 [3]. *Let the following condition be satisfied:*

$$\left(\|p_0(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]}T + \frac{1}{2}\|p_2(t)\|_{C[0,T]}T^2 + \frac{1}{3}\|a(t)\|_{C[0,T]}T^2 \right) < 1. \tag{9}$$

Then problem (7) - (8) does not have a non-trivial solution.

Along with the inverse boundary value problem (1)-(6), we consider the following auxiliary inverse boundary value problem: it is required to determine a triple $\{u(x,t), a(t), b(t)\}$ of functions $u(x,t) \in C^{2,3}(D_T),\ a(t) \in C[0,T],\ \ b(t) \in C[0,T],$ satisfying relations (1)-(3) and the following relations

$$u_x(1,t) = 0,\ 0 \le t \le T, \tag{10}$$

$$h_1'''(t) + u_{xx}(0,t) = a(t)h_1(t) + b(t)g(0,t) + f(0,t),\ \ 0 \le t \le T, \tag{11}$$

$$h_2'''(t) + u_{xx}(1,t) = a(t)h_2(t) + b(t)g(1,t) + f(1,t),\ \ 0 \le t \le T\ , \tag{12}$$

where

$$h(t) \equiv h_1(t)g(1,t) - h_2(t)g(0,t) \ne 0,\ 0 \le t \le T.$$

Theorem 1. *Let* $\varphi_i(x) \in C[0, 1]$, $p_i(t) \in C[0,T]$, $i = 0,1, 2$, $h_i(t) \in C^3[0,T]$, $i = 1,2$, $h(t) \equiv h_1(t)g(1,t) - h_2(t)g(0,t) \neq 0$, $0 \le t \le T$, $f(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = 0$, $0 \le t \le T$, $g(x,t) \in C(D_T)$, $\int_0^1 g(x,t)dx = 0$, $0 \le t \le T$, *and the following conditions of agreement are satisfied*

$$\int_0^1 \varphi_i(x)dx = 0,\ i = 0,1,2, \tag{13}$$

$$\varphi_0(0) = h_1(0) - \int_0^T p_0(t)h_1(t)dt,\ \ \varphi_1(0) = h_1'(T) - \int_0^T p_1(t)h_1(t)dt,$$

$$\varphi_2(0) = h_1''(0) - \int_0^T p_2(t)h_1(t)dt, \tag{14}$$

$$\varphi_0(1) = h_2(0) - \int_0^T p_0(t)h_2(t)dt,\ \ \varphi_1(1) = h_2'(T) - \int_0^T p_1(t)h_2(t)dt,$$

$$\varphi_2(1) = h_2''(0) - \int_0^T p_2(t)h_2(t)dt\ . \tag{15}$$

Then the following statements are true: (i) each classical solution of problem (1)-(6) is also a solution of problem (1)-(3), (10) –(12); (ii) any solution to problem (1)-(3), (10)-(12) that satisfies condition (9) is a solution to problem (1)-(6).

Proof. Let $\{u(x,t), a(t), b(t)\}$ be the solution of problem (1)-(6). Integrating equation (1) with respect to x from 0 to 1, we have

$$\frac{d^3}{dt^3}\int_0^1 u(x,t)dx + u_x(1,t) - u_x(0,t) =$$

$$= a(t)\int_0^1 u(x,t)dx + b(t)\int_0^1 g(x,t)dx + \int_0^1 f(x,t)dx,\ \ 0 \le t \le T, \tag{16}$$

whence, by conditions $\int_0^1 f(x,t)dx = 0$, $\int_0^1 g(x,t)dx = 0$, $0 \le t \le T$, and (4), implies (10).

Next, using $h_i(t) \in C^3[0,T]$, $i = 1,2$, and differentiating three times (5) and (6), respectively, we get

$$u_t(0,t)=h_1'(t),\ u_{tt}(0,t)=h_1''(t),\ u_{ttt}(0,t)=h_1'''(t),\ 0\le t\le T, \tag{17}$$

$$u_t(0,t)=h_2'(t),\ u_{tt}(1,t)=h_2''(t),\ u_{ttt}(0,t)=h_2'''(t),\ 0\le t\le T, \tag{18}$$

Substituting $x=0$ and $x=1$ into equation (1), we obtain, respectively,

$$\begin{aligned} &u_{tt}(0,t)-u_{xx}(0,t)+\alpha u_{xxxx}(0,t)-\beta u_{xxtt}(1,t)=\\ &=a(t)u(0,t)+b(t)g(0,t)+f(0,t),\ 0\le t\le T, \end{aligned} \tag{19}$$

$$\begin{aligned} &u_{tt}(1,t)-u_{xx}(1,t)+\alpha u_{xxxx}(1,t)-\beta u_{xxtt}(1,t)=\\ &=a(t)u(1,t)+b(t)g(1,t)+f(1,t),\ 0\le t\le T. \end{aligned} \tag{20}$$

From (19), taking into account (5), (17) and from (20), taking into account (6), (18), it follows, respectively, that (11) and (12) are satisfied.

Now suppose that $\{u(x,t),a(t),b(t)\}$ is a solution to problem (1)-(3), (10)-(12), and (9) is satisfied. Then from (16), taking into account (3) and (10), we obtain

$$\frac{d^3}{dt^3}\int_0^1 u(x,t)dx - a(t)\int_0^1 u(x,t)dx=0,\ 0\le t\le T. \tag{21}$$

By (2) and conditions $\int_0^1 \varphi_i(x)dx=0,\ i=0,1,2,$ we get

$$\begin{aligned} &\int_0^1 u(x,0)dx-\int_0^T p_0(t)\left(\int_0^1 u(x,t)dx\right)dt=\int_0^1\left(u(x,0)-\int_0^T p_0(t)u(x,t)dt\right)dx=\\ &\int_0^1\varphi_0(x)dx=0,\\ &\int_0^1 u_t(x,T)dx-\int_0^T p_1(t)\left(\int_0^1 u(x,t)dx\right)dt=\int_0^1\left(u_t(x,T)-\int_0^T p_0(t)u(x,t)dt\right)dx=\\ &\int_0^1\varphi_1(x)dx=0,\\ &\int_0^1 u_{tt}(x,0)dx-\int_0^T p_0(t)\left(\int_0^1 u(x,t)dx\right)dt==\int_0^1\left(u_{tt}(x,0)-\int_0^T p_0(t)u(x,t)dt\right)dx=\\ &\int_0^1\varphi_2(x)dx=0. \end{aligned} \tag{22}$$

Since, by virtue of Lemma 1, problem (21), (22) has only a trivial solution, then $\int_0^1 u(x,t)dx=0,$ i.e. conditions in (4) are satisfied.

Now, from (11), (19) and (12), (20) we get

$$\frac{d^3}{dt^3}(u(0,t)-h_1(t))=a(t)(u(0,t)-h_1(t)),\ 0\le t\le T, \tag{23}$$

$$\frac{d^3}{dt^3}(u(1,t)-h_2(t))=a(t)(u(1,t)-h_2(t)),\ \ 0\le t\le T. \tag{24}$$

By (2) and agreement conditions (14), (15), we have

$$u(0,0)-h_1(0)-\int_0^T p_0(t)(u(0,t)-h_1(t))dt=u(0,0)-\int_0^T p_0(t)u(0,t)dt-$$

$$-\left(h_1(0)-\int_0^T p_0(t)h_1(t)dt\right)=\varphi_0(0)-\left(h_1(0)-\int_0^T p_0(t)h_1(t)dt\right)=0,$$

$$u_t(0,T)-h_1'(T)-\int_0^T p_1(t)(u(0,t)-h_1(t))dt=u_t(0,T)-\int_0^T p_1(t)u(0,t)dt-$$

$$-\left(h_1'(T)-\int_0^T p_1(t)h_1(t)dt\right)=\varphi_1(0)-\left(h_1'(T)-\int_0^T p_1(t)h_1(t)dt\right)=0,$$

$$u_{tt}(0,0)-h_1''(0)-\int_0^T p_2(t)(u(0,t)-h_1(t))dt=u_{tt}(0,0)-\int_0^T p_2(t)u(0,t)dt-$$

$$-\left(h_1''(0)-\int_0^T p_2(t)h_1(t)dt\right)=\varphi_2(0)-\left(h_1''(0)-\int_0^T p_2(t)h_1(t)dt\right)=0, \tag{25}$$

$$u(1,0)-h_2(0)-\int_0^T p_0(t)(u(1,t)-h_2(t))dt=u(1,0)-\int_0^T p_0(t)u(1,t)dt-$$

$$-\left(h_2(0)-\int_0^T p_0(t)h_2(t)dt\right)=\varphi_0(1)-\left(h_2(0)-\int_0^T p_0(t)h_2(t)dt\right)=0,$$

$$u_t(1,T)-h_2'(T)-\int_0^T p_1(t)(u(1,t)-h_2(t))dt=u_t(1,T)-\int_0^T p_1(t)u(1,t)dt-$$

$$-\left(h_2'(T)-\int_0^T p_1(t)h_2(t)dt\right)=\varphi_1(1)-\left(h_2'(T)-\int_0^T p_1(t)h_2(t)dt\right)=0,$$

$$u_{tt}(1,0)-h_2''(0)-\int_0^T p_2(t)(u(1,t)-h_2(t))dt=u_{tt}(1,0)-\int_0^T p_2(t)u(1,t)dt-$$

$$-\left(h_2''(0)-\int_0^T p_2(t)h_2(t)dt\right)=\varphi_2(1)-\left(h_2''(0)-\int_0^T p_2(t)h_2(t)dt\right)=0. \tag{26}$$

From (23), (25) and (24), (26), by virtue of Lemma 1, it follows that conditions (5) and (6) are satisfied. The proof of this theorem is complete.

3. Investigation of the existence and uniqueness of a classical solution to an inverse boundary value problem

We will look for the first component $u(x,t)$ of the solution $\{u(x,t), a(t), b(t)\}$ of problem (1)-(3), (10)-(12) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \tag{27}$$

where

$$\lambda_k = k\pi,\ u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x\, dx,\ k = 0,1,2,\ \ldots,\ m_k = \begin{cases} 1, & k = 0, \\ 2, & k \geq 2, \end{cases}$$

(see, for example, [10]). Then, applying the formal scheme of the Fourier method, from (1), (2), we obtain

$$u_k'''(t) - \lambda_k^2 u_k(t) = F_k(t; u, a, b),\ 0 \leq t \leq T,\ k = 0,1,2, \ldots, \tag{28}$$

$$u_k(0) = \varphi_{0k} + \int_0^T p_0(t) u_k(t) dt,\ u_k'(T) = \varphi_{1k} + \int_0^T p_1(t) u_{ik}(t) dt,$$

$$u_k''(0) = \varphi_{2k} + \int_0^T p_2(t) u_k(t) dt,\ k = 0,1,2,\ \ldots, \tag{29}$$

where

$$F_k(t; u, a, b) = f_k(t) + a(t) u_k(t) + b(t) g_k(t),\quad f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$g_k(t) = m_k \int_0^1 g(x,t) \cos \lambda_k x dx,\ \varphi_{ik} = m_k \int_0^1 \varphi_i(x) \cos \lambda_k x dx,\ i,k = 0,1,2,\ \ldots\ldots$$

Next, from (28) and (29) we find

$$u_0(t) = \varphi_{00} + \int_0^T p_0(t) u_0(t) dt + t\left(\varphi_{10} + \int_0^T p_1(t) u_0(t) dt\right) +$$

$$+\ T\left(\frac{t}{2} - T\right)\left(\varphi_2 + \int_0^T p_2(t) u_0(t) dt\right) + \int_0^T G_0(t,\tau) F_0(\tau, u, a, b) d\tau, \tag{30}$$

$$u_k(t) = \left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}} T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}} T\right)^{-1} \left\{\left(\varphi_{0k} + \int_0^T p_0(t) u_k(t) dt\right) \times\right.$$

$$\times\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)+\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{6}\right)\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]+$$

$$+\frac{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}}{\lambda_k^{\frac{2}{3}}}\left(\varphi_{1k}+\int_0^T p_1(t)u_k(t)dt\right)\left[e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right]+$$

$$+\frac{2}{\sqrt{3}}\frac{1}{\lambda_k^{\frac{4}{3}}}\left(\varphi_{2k}+\int_0^T p_2(t)u_k(t)dt\right)\times$$

$$\times\left[\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)-\right.$$

$$\left.\left.-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]\right\}+$$

$$+\int_0^T G_k(t,\tau)F_k(\tau,u,a,b)\,d\tau,\ k=1,2,\ldots, \tag{31}$$

where

$$G_0(t,\tau)=\begin{cases}-(T-\tau)t, & t\in[0,\tau],\\ -Tt+\dfrac{t^2+\tau^2}{2}, & t\in[\tau,T],\end{cases}\qquad G_k(t,\tau)=\begin{cases}\alpha_k(t,\tau), & 0\le t\le\tau\le T,\\ \beta_k(t,\tau), & 0\le\tau\le t\le T,\end{cases}$$

$\alpha_k(t,\tau)$ and $\beta_k(t,\tau)$, $0\le\tau\le t\le T$, are determined by the formulas presented on page 1651 in the paper [3].

After substituting the expression for $u_0(t)$ from (30), $u_k(t)$, $k=1,2,\ldots$, from (31) into (27), to determine the component $u(x,t)$ of the solution to problem (1)-(3), (10)-(12), we obtain

$$u(x,t)=\varphi_{00}+\int_0^T p_0(t)u_0(t)dt+t\left(\varphi_{10}+\int_0^T p_1(t)u_0(t)dt\right)+$$

$$+T\left(\frac{t}{2}-T\right)\left(\varphi_{20}+\int_0^T p_2(t)u_0(t)dt\right)+\int_0^T G_0(t,\tau)F_0(\tau,u,a,b)d\tau+$$

$$+\sum_{i=1}^{\infty}\left\{\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left\{\left(\varphi_{0k}+\int_0^T p_0(t)u_k(t)dt\right)\times\right.\right.$$

$$\times\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)+\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{6}\right)\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]+$$

$$+\frac{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}}{\lambda_k^{\frac{2}{3}}}\left(\varphi_{1k}+\int_0^T p_1(t)u_k(t)dt\right)\left[e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right]+$$

$$+\frac{2}{\sqrt{3}}\frac{1}{\lambda_k^{\frac{4}{3}}}\left(\varphi_{2k}+\int_0^T p_2(t)u_k(t)dt\right)\times$$

$$\times\left[\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)-\right.$$

$$\left.\left.-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]\right\}+$$

$$\left.+\int_0^T G_k(t,\tau)F_k(\tau,u,a,b)d\tau\right\}\cos\lambda_k x. \tag{32}$$

Now, from (11) and (12), taking into account (27), we obtain respectively:

$$a(t)h_1(t)+b(t)g(0,t)=h_1''(t)-f(0,t)-\sum_{k=1}^{\infty}\lambda_k^2 u_k(t), \tag{33}$$

$$a(t)h_2(t)+b(t)g(1,t)=h_2''(t)-f(1,t)-\sum_{k=1}^{\infty}(-1)^k\lambda_k^2 u_k(t). \tag{34}$$

Suppose that the inequality holds

$$h(t)\equiv h_1(t)g(1,t)-h_2(t)g(0,t)\neq 0,\ 0\le t\le T.$$

Then from (33) and (34) we obtain

$$a(t)=[h(t)]^{-1}\Big\{(h_1''(t)-f(0,t))\,g(1,t)-(h_2''(t)-f(1,t))\,g(0,t)-$$
$$-\sum_{k=1}^{\infty}(g(1,t)-(-1)^k g(0,t))\,\lambda_k^2 u_k(t)\Big\}, \tag{35}$$

$$b(t)=[h(t)]^{-1}\{h_1(t)(h_2''(t)-f(1,t))-h_2(t)(h_1''(t)-f(0,t))-$$
$$-\sum_{k=1}^{\infty}((-1)^k h_1(t)-h_2(t))\,\lambda_k^2 u_k(t)\Big\}. \tag{36}$$

Substituting the expressions $u_k(t), k=1,2,\dots,$ from (31) into (35) and (36), respectively, we get

$$a(t)=[h(t)]^{-1}\{(h_1''(t)-f(0,t))\,g(1,t)-(h_2''(t)-f(1,t))\,g(0,t)-$$
$$-\sum_{i=1}^{\infty}\big(g(1,t)-(-1)^k g(0,t)\big)\lambda_k^2\left\{\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left\{\left(\varphi_{0k}+\int_0^T p_0(t)u_k(t)dt\right)\times\right.\right.$$
$$\times\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)+\right.$$
$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{6}\right)\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]+$$
$$+\frac{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}}{\lambda_k^{\frac{2}{3}}}\left(\varphi_{1k}+\int_0^T p_1(t)u_k(t)dt\right)\left[e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right]+$$

$$+\frac{2}{\sqrt{3}}\frac{1}{\lambda_k^{\frac{4}{3}}}\left(\varphi_{2k}+\int_0^T p_2\ (t)u_k(t)dt\right)\times$$

$$\times\left[\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)-\right.$$

$$\left.\left.\left.-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]\right\}+\int_0^T G_k\ (t,\tau)F_k\ (\tau,u,a,b)d\tau\right\}, \quad (37)$$

$$b(t)=\left[h(t)\right]^{-1}\left\{h_1(t)(h_2''(t)-f(1,t))-h_2(t)(h_1''(t)-f(0,t))-\right.$$

$$-\sum_{i=1}^{\infty}\left((-1)^k h_1(t)-h_2(t)\right)\lambda_k^2\left\{\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left\{\left(\varphi_{0k}+\int_0^T p_0(t)u_k(t)dt\right)\times\right.\right.$$

$$\times\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)+\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{6}\right)\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right]+$$

$$+\frac{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}}{\lambda_k^{\frac{2}{3}}}\left(\varphi_{1k}+\int_0^T p_1(t)u_k(t)dt\right)\left[e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right]+$$

$$+\frac{2}{\sqrt{3}}\frac{1}{\lambda_k^{\frac{4}{3}}}\left(\varphi_{2k}+\int_0^T p_2\ (t)u_k(t)dt\right)\times$$

$$\times\left[\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\left(e^{\lambda_k^{\frac{2}{3}}t}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\frac{\pi}{3}\right)\right)-\right.$$

$$-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\Bigg]\Bigg]\Bigg\}+\int_0^T G_k(t,\tau)F_k(\tau,u,a,b)d\tau\Bigg\}. \qquad (38)$$

Thus, the solution of problem (1)-(3), (10)-(12) is reduced to the solution of system (32), (37), (38) with respect to unknown functions $u(x,t),$, $a(t)$ and $b(t)$.

To study the question of uniqueness of the solution of problem (1)-(3), (10)-(12), the following lemma plays an important role.

Lemma 2. *Let $\{u(x,t),a(t),b(t)\}$ is an arbitrary classical solution of (1)-(3), (10)-(12). Then the functions $u_k(t), k=0,1,2,...,$ defined by the relations*

$$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x\,dx,\ k=0,1,2,\ ...\ ,$$

which satisfy the counting system (30) and (31) on $[0,T]$.

Proof. Let $\{u(x,t),a(t),b(t)\}$ is an any solution of problem (1)-(3), (10)-(12). Then multiplying both parts of equation (1) by the function $m_k\cos\lambda_k x,$ $k=0,1,2,...,$ integrating the resulting equality over x from 0 to 1 and using the relations

$$m_k\int_0^1 u_{ttt}(x,t)\cos\lambda_k xdx=\frac{d^3}{dt^3}\left(m_k\int_0^1 u(x,t)\cos\lambda_k xdx\right)=u_k'''(t),\ k=0,1,2,...,$$

$$m_k\int_0^1 u_{xx}(x,t)\cos\lambda_k xdx=-\lambda_k^2\left(m_k\int_0^1 u(x,t)\cos\lambda_k xdx\right)=-\lambda_k^2 u_k(t),\ k=0,1,2,...,$$

we are convinced that equation (28) is satisfied. Similarly, from (2) we obtain that condition (29) is satisfied.

Thus, $u_k(t), k=0,1,2,...,$ is a solution of problem (28), (29). From this it follows directly that the functions $u_k(t), k=0,1,2,...,$ satisfy the system (30), (31) on $[0,T]$. The lemma is proved.

Obviously, if $u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k xdx,\ k=0,1,2,...,$ is a solution to system (30) and (31), then the triple $\{u(x,t),a(t),b(t)\}$ of functions $u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x,$ $a(t)$ and $b(t)$ is a solution to system (32), (37), (38).

Corollary 1. *Let system (32), (37), (38) has unique solution. Then problem (1)-*

(3), (10)-(12) cannot have more than one solution, i.e. if problem (1)-(3), (10)-(12) has a solution, then it is unique.

Denote by $B_{2,T}^3$ (see [3, 4]), the space of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x,\ (x,t)\in D_T,$$

where each of the functions $u_k(t)$ is continuous on $[0,T]$ and for it the following relation holds

$$I(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty.$$

We define the norm on this space as follows:

$$\|u(x,t)\|_{B_{2,T}^3}=I(u).$$

By E_T^3 we denote the space consisting of the topological product $B_{2,T}^3\times C[0,T]\times C[0,T]$. The norm of the element $z=\{u,a,b\}\in E_T^3$ is defined by the formula

$$\|z\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

Note that $B_{2,T}^3$ and E_T^3 are Banach spaces (see, e.g., [2]).

We define in E_T^3 an operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\}=\{\sum_{k=0}^{\infty}\tilde{u}_k(t)X_k(x),\tilde{a}(t),\tilde{b}(t)\},$$

where $\tilde{u}_0(t)$, $\tilde{u}_k(t)$, $\tilde{a}(t)$ and $\tilde{b}(t)$ are determined by the right-hand sides (30), (31), (37) and (38), respectively.

Using simple transformations and direct calculations we obtain

$$\begin{aligned}\|\tilde{u}_0(t)\|_{C[0,T]}\le|\varphi_{00}|+T\|p_0(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+T\Big(|\varphi_{10}|\ +\\ +T\|p_1(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}\Big)+\frac{3}{2}T^2\Big(|\varphi_{20}|+T\|p_2(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}\Big)+\\ +3T^2\sqrt{T}\left(\int_0^T|f_0(\tau)|^2d\tau\right)^{\frac{1}{2}}+3T^2\sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T|g_0(\tau)|^2d\tau\right)^{\frac{1}{2}}+\\ +3T^3\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]},\end{aligned}\tag{39}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq$$

$$\leq 8\sqrt{15}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\varphi_{0k}|\right)^2\right)^{\frac{1}{2}} + \|p_0(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+6\sqrt{5}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{1k}|)^2\right)^{\frac{1}{2}} + \|p_1(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+8\sqrt{15}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^2|\varphi_{2k}|)^2\right)^{\frac{1}{2}} + \|p_2(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+10\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + 10\sqrt{5T}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|g_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}} +$$

$$+10\sqrt{5}\,T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{40}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t) - f(0,t))g(1,t) - (h_2''(t) - f(1,t))g(0,t)\right\|_{C[0,T]} +\right.$$

$$+\left\||g(1,t)| + |g(0,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \times$$

$$\times\left[8\sqrt{3}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{0k}|)^2\right)^{\frac{1}{2}} + \|p_0(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +\right.$$

$$+6\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{1k}|)^2\right)^{\frac{1}{2}} + \|p_1(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+8\sqrt{3}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^2|\varphi_{2k}|)^2\right)^{\frac{1}{2}} + \|p_2(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right] +$$

$$+10\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+10\sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|g_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.\left.+10\,T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}, \tag{41}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h_1(t)(h_2''(t)-f(1,t))-h_2(t)(h_1''(t)-f(0,t))\right\|_{C[0,T]}+\right.$$

$$+\left\||h_1(t)|+|h_2(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left[8\sqrt{3}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{0k}|)^2\right)^{\frac{1}{2}}+\|p_0(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+\right.$$

$$+6\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{1k}|)^2\right)^{\frac{1}{2}}+\|p_1(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+$$

$$+8\sqrt{3}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^2|\varphi_{2k}|)^2\right)^{\frac{1}{2}}+\|p_2(t)\|_{C[0,T]}T\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]+$$

$$+10\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+10\sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|g_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.\left.+10\,T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{42}$$

Assume that the data of problems (1)-(3), (10)-(12) satisfy the following conditions:

1. $\varphi_i(x)\in C^2[0,1],\ \varphi_i'''(x)\in L_2(0,1),\ \varphi_i'(0)=\varphi_i'(1)=0,\ i=0,1,$
2. $\varphi_2(x)\in C^1[0,1],\ \varphi_2''(x)\in L_2(0,1),\ \varphi_2'(0)=\varphi_2'(1)=0,$.
3. $f(x,t),\ f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),\ \ f_x(0,t)=f_x(1,t)=0,\ 0\leq t\leq T,$

4. $g(x,t), g_x(x,t) \in C(D_T),\ g_{xx}(x,t) \in L_2(D_T),\ g_x(0,t) = g_x(1,t) = 0,\ 0 \le t \le T,$

5. $p_i(t) \in C[0,T],\ i = 0,1,2,\ h_i(t) \in C^3[0,T],\ i = 1,2,\ h(t) \equiv h_1(t)g(1,t) - h_2(t)g(0,t) \ne 0,$
$0 \le t \le T.$

Then, by virtue of (39), (40), (41) and (42), we have, respectively,

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le A_1(T) +$$
$$+ B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_1(T)\|u(x,t)\|_{B^3_{2,T}} + D_1(T)\|b(t)\|_{C[0,T]}, \tag{43}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le A_2(T) +$$
$$+ B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\|u(x,t)\|_{B^3_{2,T}} + D_2(T)\|b(t)\|_{C[0,T]}, \tag{44}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_3(T) +$$
$$+ B_3(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_3(T)\|u(x,t)\|_{B^3_{2,T}} + D_3(T)\|b(t)\|_{C[0,T]}, \tag{45}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le A_4(T) +$$
$$+ B_4(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_4(T)\|u(x,t)\|_{B^3_{2,T}} + D_4(T)\|b(t)\|_{C[0,T]}, \tag{46}$$

where

$$A_1(T) = \|\varphi_0(x)\|_{L_2(0,1)} + T\|\varphi_1(x)\|_{L_2(0,1)} + \frac{3}{2}T^2\|\varphi_2(x)\|_{L_2(0,1)} + 3T^2\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = 3T^3,\ C_1(T) = T\|p_0(t)\|_{C[0,T]} + T^2\|p_1(t)\|_{C[0,T]} + \frac{3}{2}T^3\|p_2(t)\|_{C[0,T]},$$

$$D_1(T) = 3T^2\sqrt{T}\|g(x,t)\|_{L_2(D_T)},$$

$$A_2(T) = 8\sqrt{15}\|\varphi_0'''(x)\|_{L_2(0,1)} +$$
$$+ 6\sqrt{5}\|\varphi_1'''(x)\|_{L_2(0,1)} + 8\sqrt{15}\|\varphi_2''(x)\|_{L_2(0,1)} + 10\sqrt{5T}\|f_{xx}(x,t)\|_{L_2(D_T)},$$

$$B_2(T) = 10\sqrt{5}T,,$$

$$C_2(T) = 8\sqrt{15}\ T\|p_0(t)\|_{C[0,T]} + 6\sqrt{5}\ T\|p_1(t)\|_{C[0,T]} + 8\sqrt{15}\ T\|p_2(t)\|_{C[0,T]},$$

$$D_1(T) = 10\sqrt{5T}\|g_{xx}(x,t)\|_{L_2(D_T)},$$

$$A_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_1''(t) - f(0,t)\right)g(1,t) - \left(h_2''(t) - f(1,t)\right)g(0,t)\right\|_{C[0,T]} +\right.$$

$$+\left\| |g(1,t)| + |g(0,t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times$$

$$\times \left[8\sqrt{3} \|\varphi_0'''(x)\|_{L_2(0,1)} + 6\|\varphi_1'''(x)\|_{L_2(0,1)} + 8\sqrt{3}\|\varphi_2''(x)\|_{L_2(0,1)} + 20\sqrt{T}\, \| f_{xx}(x,t)\|_{L_2(D_T)} \right] \Bigg\},$$

$$B_3(T) = 10 \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| |g(1,t)| + |g(0,t)| \right\|_{C[0,T]} T,$$

$$C_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| |g(1,t)| + |g(0,t)| \right\|_{C[0,T]} \times$$

$$\times T \left(8\sqrt{3} \|p_0(t)\|_{C[0,T]} + 6\|p_1(t)\|_{C[0,T]} + 8\sqrt{3}\|p_2(t)\|_{C[0,T]} \right),$$

$$D_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\| |g(1,t)| + |g(0,t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} 20\sqrt{T} \|g_{xx}(x,t)\|_{L_2(D_T)},$$

$$A_4(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \Big\{ \left\| h_1(t)\left(h_2''(t) - f(1,t)\right) - h_2(t)\left(h_1''(t) - f(0,t)\right) \right\|_{C[0,T]} +$$

$$+\left\| |h_1(t)| + |h_2(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times$$

$$\times \left[8\sqrt{3} \|\varphi_0'''(x)\|_{L_2(0,1)} + 6\|\varphi_1'''(x)\|_{L_2(0,1)} + 8\sqrt{3}\|\varphi_2''(x)\|_{L_2(0,1)} + 20\sqrt{T}\, \| f_{xx}(x,t)\|_{L_2(D_T)} \right] \Bigg\},$$

$$B_4(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\| |h_1(t)| + |h_2(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} T$$

$$C_4(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\| |h_1(t)| + |h_2(t)| \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}\frac{1}{2}} \times$$

$$\times T \left(8\sqrt{3} \|p_0(t)\|_{C[0,T]} + 6\|p_1(t)\|_{C[0,T]} + 8\sqrt{3}\|p_2(t)\|_{C[0,T]} \right),$$

$$D_4(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\||h_1(t)|+|h_2(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}20\sqrt{T}\left\|g_{xx}(x,t)\right\|_{L_2(D_T)}.$$

It follows from (43)-(46) that

$$\|\tilde{u}(x,t)\|_{B_{2,T}^6}+\|\tilde{a}(t)\|_{C[0,T]}+\|\tilde{b}(t)\|_{C[0,T]}\le A(T)+$$
$$+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^5}+C(T)\|u(x,t)\|_{B_{2,T}^5}+D(T)\|b(t)\|_{C[0,T]}, \qquad (47)$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T)+A_4(T),\ B(T)=B_1(T)+B_2(T)+B_3(T)+B_4(T),$$
$$C(T)=C_1(T)+C_2(T)+C_3(T)+C_4(T),\ D(T)=D_1(T)+D_2(T)+D_3(T)+D_4(T).$$

Let $K_R=\{z\in E_T^3:\|z\|_{E_T^3}\le R\}$.

The main result of this paper is the following theorem.

Theorem 2. *Let conditions 1^0–5^0 be satisfied and the following inequality holds*

$$(B(T)(A(T)+2)+C(T)+D(T))(A(T)+2)<1. \qquad (48)$$

Then problem (1)-(3), (10)-(12) has a unique solution in the ball $K_R\subset E_T^3$ for $R=A(T)+2$.

Proof. Consider the following equation

$$z=\Phi z,\ z=\{u,a,b\}\in E_T^3, \qquad (49)$$

where the components $\Phi_i, i=1,2,3,$ of the operator Φ are determined by the right-hand sides of equations (32), (37) and (38).

Let $R\le A(T)+2$. We consider the operator Φ in the ball $K_R\subset E_T^3$. Similarly to (47), it is easy to show that for any $z,z_1,z_2\in K_R$ the following estimates hold:

$$\|\Phi z\|_{E_T^3}\le A(T)+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+C(T)\|u(x,t)\|_{B_{2,T}^3}+D(T)\|b(t)\|_{C[0,T]}\le$$
$$\le A(T)+B(T)(A(T)+2)^2+C(T)(A(T)+2)+D(T)(A(T)+2), \qquad (50)$$
$$\|\Phi z_1-\Phi z_2\|_{E_T^3}\le B(T)R(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3})+$$
$$+C(T)\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}+D(T)\|b_1(t)-b_2(t)\|_{C[0,T]}. \qquad (51)$$

Then, by virtue of (48), it follows from (50) and (51) that the operator Φ on the set $K_R\subset E_T^3$ satisfies the conditions of the principle of contraction mapping. Therefore, this operator in the ball K_R has a unique fixed point $z=\{u,a,b\}$, which is a solution of equation (49), i.e. $z=\{u,a,b\}$ is the only solution of system

(32), (37), (38) in the ball K_R.

The function $u(x,t)$ as an element of the space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T. Moreover, from (28) we get

$$\|u_0'''(t)\|_{C[0,T]} \le \|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]} + 2\left\| \|f(x,t)\|_{C[0,T]} \right\|_{L_2(0,1)},$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'''(t)\|_{C[0,T]})^2\right)^{1/2} \le \sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{1/2} +$$

$$\sqrt{2}\left\| \|a(t)u_x(x,t) + f_x(x,t)\|_{C[0,T]} \right\|_{L_2(0,1)},$$

which implies that $u_{ttt}(x,t) \in C(D_T)$.

It is easy to verify that equation (1) and conditions (2), (3), (10), (11) and (12) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution to problem (1)-(3), (10)-(12). By Corollary 2 it is unique. The proof of this theorem is complete.

Theorem 1, Theorem 2 and Corollary 1 implies the unique solvability of the original problem (1)-(6).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied, and also let the conditions*

$$\int_0^1 f(x,t)dx = 0,\ \int_0^1 g(x,t)dx = 0,\ 0 \le t \le T,$$

$$\left(\|p_0(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]}T + \frac{1}{2}\|p_2(t)\|_{C[0,T]}T^2 + \frac{1}{3}(A(T)+2)T^2 \right) < 1,$$

and the matching conditions (13)-(15) be satisfied. Then problem (1)-(5) has a unique classical solution in the ball K_R for $R = A(T)+2$.

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