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Analytical solution of the Klein-Fock-Gordon equation for some spherically symmetrical potentials

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Abstract

In the presented work, the Klein-Fock-Gordon (KFG) equation is solved by the Nikiforov-Uvarov (NU) method for some spherically symmetric potentials: a linear combination of Hulthen and Yukawa class potentials, as well as a linear combination of Manning-Rosen plus Yukawa class potentials, by reducing it to a hypergeometric equation in quantum mechanics. Finding optimal solutions of wave equations for arbitrary values of theorbital quantum number l≠0 in such potential fields is considered one of the urgent problems of theoretical physics. As a result, analytical expressions for the energy spectrum, eigenfunction and normalization constants at arbitrary values of orbital quantum number have been found from the solution of the KFG equation. It has been shown that the energy spectrum and eigenfunction depend on the choice of the orbital quantum number l. Also, the energy levels and corresponding normalized eigenfunctions are represented as a recursion relation in terms of the Jacobi polynomials for arbitrary l states. Futhermore, a closed form for the normalization constant of the wave functions is found.

Keywords: Hulthen potential, Manning-Rosen potential, Yukawa potential, Nikiforov-Uvarov method
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1. Introduction

Exactly solvable problems for quantum systems have long been a subject of intense study in many branches of quantum physics. The primary motivation for finding analytical solutions is that the wave function contains all the requisite infor-

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mation for the full description of a quantum system [1–6]. In physics, especially the relativ- istic quantum mechanical applications to particle and nuclear physics, the relativistic wave equations predict particles reactions at high energies [5–7]. The analytical solution of the Klein-Fock-Gordon (KFG) equation with physical potentials plays a central role in relativistic quantum mechanics since this wave equation perfectly defines the spinless pseudoscalar pions and Higgs boson. There are many techniques to solve the wave equations with potentials in the relativistic and also non-relativistic cases. The following are some of them: Nikiforov-Uvarov (NU) method [8], supersymmetry QM (SUSYQM) [9], shifted 1/N expansion method [10,11], asymptotic iteration method, Hartree-Fock method, the path integral method, factorization and perturba- tion theory. Among them, the NU and SUSYQM methods have received great interest. By using these two techniques, many works have been conducted to obtain either exact or approximate solutions of the KG equation with some well-known potentials as follows: Manning-Rosen Potential, Yukawa potential, Hulthen Potential, generalized Hulthen potential, Kratzer Potential, Wood-Saxon Potential and Deng-Fan molecular potentials. Similar studies have been conducted for the case of combined potentials: Manning–Rosen plus Hulthén potential, Hulthén plus a Ring-Shaped like potential, Hulthén plus Yukawa potential and references therein. In particularly, most of them are based on the solutions of the KG equation with equal and unequal vector and scalar potential energies.

The Hulthén potential is defined as follows [12]:

$$V_H(r) = \frac{-Ze^2\delta e^{-\delta r}}{(1-e^{-\delta r})}, \tag{1}$$

where *Z* and *a* are the atomic number and the screening parameter, respectively. They determine the range for the Hulthén potential [12]. The Manning-Rosen potential is widely used in mathematical modeling of oscillations and vibrations of diatomic molecules. It is also used in the construction of suitable models for the mathematical description of other physical phenomena, as follows[13]:

$$V_{MR}(r) = \frac{\hbar^2}{2\mu\delta^2}\left[\frac{\alpha(\alpha-1)e^{-2r/\delta}}{(1-e^{-r/\delta})^2} - \frac{Ae^{-r/\delta}}{1-e^{-r/\delta}}\right], \tag{2}$$

where A and α are dimensionless constants, and δ is a screening parameter. The Yukawa potential was proposed in 1935 as an operative potential to describe the strong inter- actions between nucleons [14]. It takes the following form:

$$V_Y(r) = -\frac{V_0 e^{-\delta r}}{r}, \tag{3}$$

where V_0 determines the strength of the interaction.

2. Research method

Two different types of potentials can be introduced into the KFG equation since it contains two components: the scalar rest mass and the four-vector linear momentum operator. These are a scalar potential $S(r)$ (which is introduced via scalar coupling) and a vector potential $V(r)$ (which is introduced via minimal coupling). In the spherical coordinates system, the KFG equation with scalar potential and vector potential is formulated by

$$[-\nabla^2 + (M + S(r)^2]\psi(r,\theta,\varphi) = [E - V(r)^2]\psi(r,\theta,\varphi), \tag{4}$$

where *M* is the rest mass of a scalar particle and *E* denotes the relativistic energy of the system. The solution to equation (4) in the stationary case in a spherical coordinate system is given by:

$$\psi(r,\theta,\varphi) = \frac{\chi(r)}{r}\theta(\theta)e^{im\varphi}; m = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{5}$$

Considering expression (5) in equation (4), let us write the radial KFG equation in the following form:

$$\chi''(r) + \Big[(E^2 - M^2) - 2(MS(r)) + EV(r)) +$$

$$+ (V^2(r) - S^2(r)) - \frac{l(l+1)}{r^2}\Big]\chi(r) = 0. \tag{6}$$

Our goal is to reduce equation (6) to the form of a hypergeometric equation in the form shown below:

$$\chi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\chi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\chi(s) = 0. \tag{7}$$

We have noted that it is impossible to solve the KFQ equation exactly for arbitrary *l*. Therefore, in order to solve the equation analytically, we apply the following approximation to the products of $1/r$ in the Yukawa potential and 2 in the centrifugal potential $1/r^2$:

$$\frac{1}{r} = \frac{2e^{-\delta r}}{1 - e^{-2\delta r}}, \quad \frac{1}{r^2} = \frac{4\delta^2 e^{-2\delta r}}{(1 - e^{-2\delta r})^2}. \tag{8}$$

This approximation is the one proposed by Green and Aldrich [15]. This approximation can only be applied when the $\delta r << 1$ condition is satisfied [16, 17]. Using this approximation, the radial KFQ equation can be solved analytically for the $l \neq 0$ case. Then we can write the Yukawa potential as follows:

$$V(r) = \frac{2\delta V_0 e^{-\delta r}}{1 - e^{-2\delta r}}. \tag{9}$$

Let us consider scalar and vector potential forms for the general Hulthén and Yukawa potentials as follows:

$$V(r) = -\frac{(V_0 + V_0^{'})e^{-\delta r}}{1 - e^{-2\delta r}}, \ \ S(r) = -\frac{(S_0 + S_{0_0}^{'})e^{-\delta r}}{1 - e^{-2\delta r}}. \tag{10}$$

Let's introduce the following substitutions:

$$\varepsilon = -\frac{\sqrt{M^2 - E^2}}{2\delta}, \propto = -\frac{\sqrt{2EV_0 + 2MS_0}}{2\delta},$$

$$\beta = -\frac{\sqrt{2EV_0^{'} + 2MS_0^{'}}}{2\delta}, \ \ \gamma = -\frac{\sqrt{S_0^2 + V_0^2}}{2\delta}, \ \ \rho = -\frac{\sqrt{S_0^{'2} + V_0^{'2}}}{2\delta}. \tag{11}$$

If we introduce the variable $s = e^{-2\delta r}$, we obtain:

$$\frac{d}{dr} = \frac{ds}{dr}\frac{dr}{ds} = -2\delta s\frac{d}{ds},$$

$$\frac{d^2}{dr^2} = 4\delta^2 s\frac{d}{ds} + 4\delta^2 s^2\frac{d^2}{ds^2}. \tag{12}$$

Considering these expressions in equation (6):

$$\chi''(s) + \frac{1-s}{s(1-s)}\chi'(s) + \frac{1}{s^2(1-s)^2}[-\varepsilon^2(1-s)^2 + \alpha^2 s(1-s) +$$

$$+\beta^2 s(1-s) + \gamma^2 s^2 - \rho^2 s^2 - l(l+1)s]\chi(s) = 0. \tag{13}$$

The resulting equation is a hypergeometric type equation.

Thus, the NU (Nikiforov-Uvarov) method can be easily used by reducing the KFQ equation to a hypergeometric type equation in the form (13). First, let's compare equation (13) with the hypergeometric equation (7). At this point, we obtain the following expressions for the coefficients of polynomials $\bar{\tau}$, σ, and $\bar{\sigma}$:

$$\bar{\tau} = 1 - s, \ \ \sigma = s(1-s),$$

$$\bar{\sigma} = -\varepsilon^2(1-s^2) + \alpha^2 s(1-s) + \beta^2 s(1-s) + \gamma^2 s^2 - \rho^2 s^2 - l(l+1)s. \tag{14}$$

Here, let's factorize the radial function $\chi(s)$ as follows:

$$\chi(s) = \phi(s)y(s). \quad ()$$

According to this method, the function $\phi(s)$ satisfies the following condition:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}. \tag{15}$$

If we use expressions (15), we can calculate the function:

$$\pi(s) = \frac{\sigma'(s) - \bar{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k\sigma(s)} =$$

$$= -\frac{s}{2} \pm \sqrt{s^2 - [a - k] - s[b - k] + c}. \tag{16}$$

Here

$$a = \frac{1}{4} + \varepsilon^2 + \alpha^2 + \beta^2 - \gamma^2 + \rho^2,$$

$$b = 2\varepsilon^2 + \alpha^2 + \beta^2 - l(l+1),$$

$$a = \frac{1}{4} + \varepsilon^2 + \alpha^2 + \beta^2 - \gamma^2 + \rho^2,$$

$$c = \varepsilon^2. \tag{17}$$

First, let's write the discriminant of the quadratic expression under the root:

$$D = (b - k)^2 - 4c(a - k) = k^2 - 2bk + b^2 - 4ac + 4ck.$$

From condition $D = 0$

$$k^2 - (4c - 2b)k + (b^2 - 4ac) = 0. \tag{18}$$

Solving equation (18) with respect to k, we obtain:

$$k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a - b)} \tag{19}$$

on the other hand

$$s^2(a - k) - s(b - k) + c = s^2\left(a - b + 2c - 2\sqrt{c^2 + c(a - b)}\right) -$$

$$-s\left(2c - 2\sqrt{c^2 + c(a - b)}\right) + c = (As - B)^2. \tag{20}$$

From this, we obtain:

$$A^2 = a - b + 2c - 2\sqrt{c^2 + c(a - b)},$$

$$2AB = 2c - 2\sqrt{c^2 + c(a - b)}, \qquad ()$$

$$B^2 = c,$$

$$A = \sqrt{c} - \sqrt{c+a-b} \qquad (21)$$

We can find four possible functions for the function $\pi(s)$:

$$\pi(s) = -\frac{s}{2} \pm$$

$$\pm \begin{cases} \left(\sqrt{c} - \sqrt{c+a-b}\right)s - \sqrt{c}, k = (b-2c) + 2\sqrt{c^2 + c(a-b)} \\ \left(\sqrt{c} + \sqrt{c+a-b}\right)s - \sqrt{c}, k = (b-2c) - 2\sqrt{c^2 + c(a-b)} \end{cases} \qquad (22)$$

According to the Nikiforov-Uvarov method, we choose one of the four possible forms of the polynomial $\pi(s)$ such that the derivative of the function $\tau(s)$ for this form polynomial is negative and the root is located in the interval $(0, 1)$.

The specific value of energy is found directly from the expression of parameter λ, that is, it is found from the condition that the two equivalent expressions of parameter λ are equal to each other:

$$\lambda = (b-2c) - 2\sqrt{c^2 + c(a-b)} - \frac{1}{2} - \left[\sqrt{c} + \sqrt{c+a-b}\right]. \qquad (23)$$

For given a non-negative integers n, an equation of hypergeometric type has only and only a unique polynomial solution of degree n:

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad (n = 0, 1, 2, \dots).$$

Here

$$\tau(s) = (1-s) + 2\pi(s) = 1 - s\left(2 + 2\left(\sqrt{c} + \sqrt{c+a-b}\right)\right) + 2\sqrt{c},$$

$$\tau'(s) = -\left(2 + 2\left(\sqrt{c} + \sqrt{c+a-b}\right)\right),$$

then

$$\lambda_n = n\left[2 + 2(\sqrt{c} + \sqrt{c+a-b})\right] - \frac{n(n-1)}{2} \cdot (-2) =$$

$$= 2n\left[1 + (\sqrt{c} + \sqrt{c+a-b})\right] + n(n-1). \qquad (24)$$

Equating expressions (23) and (24), we obtain:

$$(b-2c) - 2\sqrt{c^2 + c(a-b)} - 1/2 - \left[\sqrt{c} + \sqrt{c+a-b}\right] =$$

$$= 2n\left[1 + \left(\sqrt{c} + +\sqrt{c+a-b}\right)\right] + n(n-1),$$

$$\sqrt{c} = \varepsilon, \tag{25}$$

$$\sqrt{c+a-b} = \sqrt{1/4 - \gamma^2 + \rho^2 + l(l+1)},$$

$$(b-2c) = \alpha^2 + \beta^2 - l(l+1),$$

$$\varepsilon = \frac{\alpha^2 + \beta^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4} - \gamma^2 + \rho^2 - l(l+1)}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \gamma^2 + \rho^2 - l(l+1)}} \tag{26}$$

In the case of $V(r) \neq S(r)$, we can write the equation to calculate the numerical value of the energy spectrum in the following form:

$$M^2 - E^2 =$$

$$= \left[\frac{\alpha^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4} + \gamma^2 + \rho^2 + l(l+1)} - n(n+1)}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \gamma^2 + \rho^2 - l(l+1)}} \cdot \delta\right]^2 \tag{27}$$

The KFQ equation has been studied in detail for the case $V(r) = S(r)$. Here, with a vector potential of 2 times, the KFQ equation transforms into the Schrödinger equation. The result obtained is completely identical to the expression obtained in [18] – this research work.

Now, to find the eigenfunction of a relativistic particle of mass M moving in a Hulthén plus Yukawa potential field by applying the Nikifarov-Uvarov method, let us factorize the function $\chi(s)$ as follows:

$$\chi(s) = \phi(s) y(s), \tag{28}$$

then for functions $\phi(s)$ we get:

$$\phi(s) = s^{\varepsilon} \cdot (1-s)^{\frac{1}{2} + \sqrt{\frac{1}{4} - \gamma^2 + \rho^2 + l(l+1)}}. \tag{29}$$

İt is necessary to find the function $\rho(s)$ first. $\rho(s)$ weight functions are found from solving the Pearson differential equation. The Pearson differential equation is defined as follows:

$$(\sigma\rho)' = \tau\rho. \tag{30}$$

Let's solve this equation:

$$\rho(s) = s^{2\sqrt{s}} \cdot (1-s)^{2\sqrt{c+a-b}} = s^{2\varepsilon}(1-s)^{2\sqrt{1/4-\gamma^2+\rho^2+l(l+1)}}. \quad (31)$$

The function $y_n(s)$, which is a component of the spinor function $\chi(s)$, is determined by the Rodriques formula in the following way:

$$y_n(s) = C_n P_n^{\,2\varepsilon, 2\sqrt{1/4-\gamma^2+\rho^2+l(l+1)}}(s). \quad (32)$$

Thus, we get the division expression for the radial function $\chi(s)$:

$$\chi(s) = C_n s^{\varepsilon}(1-s)^{\left(1/2+\sqrt{1/4-\gamma^2+\rho^2+l(l+1)}\right)} P_n^{\,2\varepsilon, 2\sqrt{1/4-\gamma^2+\rho^2+l(l+1)}}(s). \quad (33)$$

If we express the function $\chi(s)$, as a hypergeometric function, we get the following expression:

$$\chi_n(s) = C_n s^{\sqrt{c}}(1-s)^K \frac{\Gamma(n+2\sqrt{c}+1)}{n!\,\Gamma(2\sqrt{c}+1)} \times$$

$$\times\ {}_2F_1\left(-n, 2\sqrt{c}+2K+n, 1+2\sqrt{c}, s\right). \quad (34)$$

Here

$$K = 1/2 + \sqrt{1/4+\gamma^2+\rho^2+l(l+1)}. \quad (35)$$

The normalization constant C_n is found from the normalization condition and is as follows:

$$C_n = \sqrt{\frac{2\delta \cdot n!\,(n+K+\sqrt{c})\Gamma(2\sqrt{c+1})\Gamma(n+2\sqrt{c}+K-1}{(n_r+K)\Gamma(n_r+2K)\Gamma(2\sqrt{c})}}. \quad (36)$$

It can be seen from the Fig. 1 that as the radial quantum number increases, the energy of the system increases more rapidly and the connection of the system with the potential center weakens, that is, the associated states of the system weaken.

Similarly, if we consider the sum of the Manning-Rosen potential and the Yukawa potential in the KFG equation, then we obtain the following equation to calculate the numerical value of the energy spectrum:

$$M^2 - E^2 =$$

$$= \left[\frac{\beta^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4}+\alpha^2+l(l+1)}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha^2+l(l+1)}} \cdot \delta\right]^2. \quad (37)$$

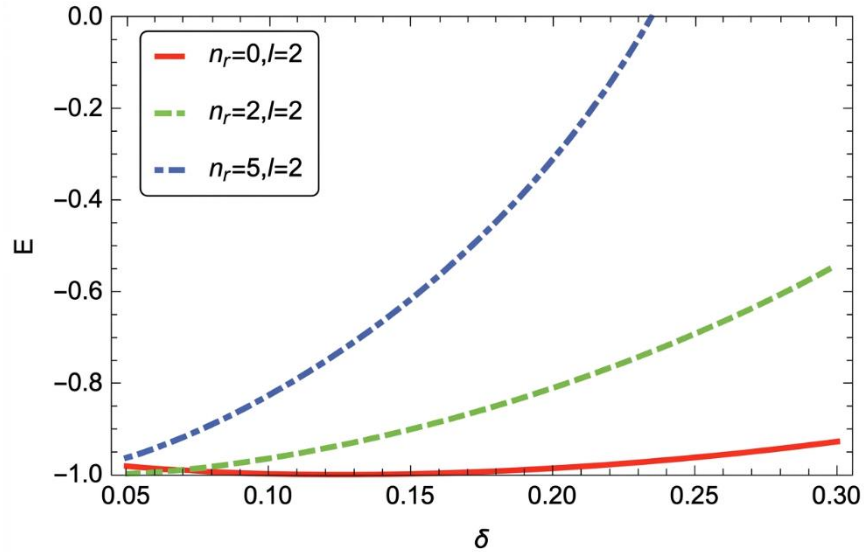

Fig. 1. Dependence of energy on the screening parameter at fixed values of radial and orbital quantum numbers.

here the following new parameters have been introduced:

$$\varepsilon = \frac{\sqrt{M^2 - E^2}}{2\delta}, \quad \alpha = \frac{\sqrt{2(M+E)V_{014}}}{2\delta}, \quad \beta = \frac{\sqrt{2(M+E)V_{023}}}{2\delta} > 0.$$

For the radial function $\boldsymbol{\chi(s)}$ we get:

$$\chi(s) \ = C_n s^{\varepsilon} (1-s)^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\alpha^2+\rho^2+l(l+1)}\right)} P_n^{2\varepsilon, 2\sqrt{\frac{1}{4}+\alpha^2+l(l+1)}}(s) \tag{38}$$

The function $\boldsymbol{\chi(s)}$ can be expressed as a hypergeometric function as follows:

$$\chi_n(s) \ = C_n s^{\sqrt{c}} (1-s)^K \frac{\Gamma(n+2\sqrt{c}+1)}{n!\,\Gamma(2\sqrt{c}+1)} \times$$

$$\times \ {}_2F_1(-n, 2\sqrt{c}+2K+n, 1+2\sqrt{c}, s) \tag{39}$$

Here

$$K = \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}$$

The normalization constant C_n is found from the normalization condition and is as follows:

$$C_n = \sqrt{\frac{2\delta \cdot n!\,(n + K + \sqrt{c})\Gamma(2\sqrt{c+1})\Gamma(n + 2\sqrt{c} + 2K)}{(n_r + K)\Gamma(n_r + 2K)\Gamma(2\sqrt{c})\Gamma(n + 2\sqrt{c} + 1)}}. \tag{40}$$

Now let's look at some special cases.

1. If $V_0 = 0, S_0 = 0$ and $V'_0 \neq S'_0$ then we directly obtain the energy spectrum for the Yukawa potential. The energy spectrum is defined as follows. here $\alpha = 0$ and $\gamma = V'_0/2\delta$, $\rho = V'_0/2\delta$. Then equatıon (27) ıs as follows [19]:

$$M^2 - E^2 =$$

$$= \left[\frac{\beta^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4} - \gamma^2 + \rho^2 + l(l+1)}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \gamma^2 + \rho^2 + l(l+1)}} \cdot \delta\right]^2$$

The specific function of the system is defined by the following expression:

$$\chi(s) = C_n s^{\varepsilon}(1-s)^K P_n^{(2\varepsilon, 2K-1)}(1-2s).$$

2. If $\boldsymbol{V_0 = V'_0 = 0}$, then we directly obtain the energy spectrum for the Manning-Rosen potential. The energy spectrum is defined as follows:

$$\boldsymbol{M^2 - E^2 =}$$

$$= \left[\frac{\boldsymbol{\gamma^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}}}{\boldsymbol{n + \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}}} \cdot \boldsymbol{\delta}\right]^2$$

Here

$$\gamma = \frac{\sqrt{2(M+E)V_{02}}}{2\delta} = \sqrt{\frac{A(M+E)}{M}}.$$

The specific function of the system is defined by the following expression:

$$\chi(s) = C_n s^{\frac{\sqrt{M^2-E^2}}{2\delta}}(1-s)^{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}\right)} \times$$

$$\times P^{\left(\frac{\sqrt{M^2-E^2}}{\delta}, 2\sqrt{\frac{1}{4} + \alpha^2 + l(l+1)}\right)}(s),$$

3. If $\boldsymbol{V_{01} = 0, V_{02} = 0}$ and, $\boldsymbol{V_{04} = 0}$ then $\boldsymbol{\eta = 1}$ and $\boldsymbol{A = V'_0 = 0}$ are obtained and we directly obtain the energy spectrum for the central Yukawa potential [20]:

$$M^2 - E^2 = \left[\frac{\xi^2 - l(l+1) - n(n+1) - \frac{1}{2} - (2n+1)\sqrt{\frac{1}{4} + l(l+1)}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}} \cdot \delta \right]^2 .$$

Here $\boldsymbol{\xi}$ is defined by the following expression:

$$\xi = \frac{\sqrt{4\delta V_0 (M+E)}}{2\delta},$$

The specific function of the system is defined by the following expression:

$$\chi(s) = C_n s^{\frac{\sqrt{M^2-E^2}}{2\delta}} (1-s)^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}\right)} P^{\left(\frac{\sqrt{M^2-E^2}}{\delta}, 2\sqrt{\frac{1}{4}+l(l+1)}\right)}(s).$$

4. Conclusion

Focusing on an improved approximation scheme, we present how to treat the centrifugal and the Coulombic behavior terms and then to obtain the bound state solutions of the Klein-Gordon (KG) equation with the Hulthén plus a Yukawa and Manning-Rosen plus a Yukawa potentials. By means of the Nikiforov-Uvarov (NU) methods, we present the energy spectrum and the corresponding radial wave functions in terms of the hypergeometric functions. Several special cases for the potentials which are useful for other physical systems are also discussed. These are consistent with those results in previous works. We obtain that the energy level *E* is sensitive to the potential parameter *δ* at fixed values of other parameters.

References

[1] Garavelli, S.L.; Oliveira, F.A. Analytical solution for a Yukawa-type potential. *Phys. Rev. Lett.* **1991**, *66*, 1310–1313, DOI.

[2] Greiner, W. *Relativistics Quantum Mechanics*, 3rd ed.; Springer: Berlin, Germany, 2000.

[3] Bagrov, V.G.; Gitman, D.M. *Exact Solutions of Relativistic Wave Equations*; Kluwer Academic Publishers: Dordrecht, Netherlands, 1990.

[4] Garavelli, S.L.; Oliveira, F.A. Title of the article. *Phys. Rev. Lett.* **1991**, *66*, 1310.

[5] Boivin, L.; Kartner, F.X.; Haus, H.A. Title of the article. *Phys. Rev. Lett.* **1994**, *73*, 240.

[6] Bialynicki-Birula, I. Title of the article. *Phys. Rev. Lett.* **2004**, *93*, 020402.

[7] Belic, M.; Petrovic, N.; Zhong, W.P.; Xie, R.H.; Chen, G. Title of the article. *Phys. Rev. Lett.* **2008**, *101*, 123904.

[8] Nikiforov, A.F.; Uvarov, V.B. *Special Functions of Mathematical Physics*; Birkhäuser: Basel, Switzerland, 1988.

[9] Cooper, F.; Khare, A.; Sukhatme, U. *Supersymmetry in Quantum Mechanics*; World Scientific: Singapore, 2001.

[10] Lütfüoğlu, B.C.; Lipovský, J.; Kříž, J. Scattering of Klein-Gordon particles in the background of mixed scalar-vector generalized symmetric Woods-Saxon potential. *Eur. Phys. J. Plus* **2018**, *133*, No. 1.

[11] Badalov, V.H.; Ahmadov, H.I.; Badalov, S.V. Any l-state analytical solutions of the Klein–Gordon equation for the Woods–Saxon potential. *Int. J. Mod. Phys. E* **2012**, *19*, 1463–1475.

[12] Hulthén, L. Arkiv foer Matematik, Astronomi, och Fysik. *Ark. Mat. Astron. Fys.* **1942**, *26*, 5.

[13] Manning, M.F.; Rosen, N. Title of the article. *Phys. Rev.* **1933**, *44*, 953.

[14] Yukawa, H. On the Interaction of Elementary Particles. I. *Proc. Phys. Math. Soc. Jpn.* **1935**, *17*, 48–57.

[15] Greene, R.L.; Aldrich, C. Variational wave functions for a screened Coulomb potential. *Phys. Rev. A* **1976**, *14*, No. 6.

[16] Qiang, W.-C.; Dong, S.-H. The Manning–Rosen potential studied by a new approximate scheme to the centrifugal term. *Phys. Scr.* **2009**, *79*, 045004.

[17] Wei, G.-F.; Dong, S.-H. Approximately analytical solutions of the Manning-Rosen potential with the spin-orbit coupling term and spin symmetry. *Phys. Lett. A* **2008**, *373*, 49–53.

[18] Saad, N. The Klein-Gordon equation with a generalized Hulthen potential in D-dimensions. *Phys. Scr.* **2007**, *76*, No. 6.

[19] Hamzavi, M.; Ikhdair, S.M.; Thylwe, K.E. Spinless particles in the field of unequal scalar – vector Yukawa potentials. *Chin. Phys. B* **2013**, *22*, 040301.

[20] Wang, Z.; Long, Z.W.; Long, C.Y. Analytical solutions of position-dependent mass Klein-Gordon equation for unequal scalar and vector Yukawa potentials. *Indian J. Phys.* **2015**, *89*, 1059–1064.