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# ON THE REPRESENTABILITY OF A SMOOTH TRIVARIATE FUNCTION BY SUMS OF GENERALIZED RIDGE FUNCTIONS

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**Abstract**

In this paper, we find a condition under which the smooth trivariate function can be represented by sums of generalized ridge functions and give a partial solution of the problem, posted by A.Pinkus in his monograph "Ridge Functions".

*Keywords:* ridge function, smooth trivariate function, representation, increment, generalized ridge function.

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## 1. Introduction

A ridge function is any multivariate function $F : R^n \to R$ of the form

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$$F(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}) = f(a_1 x_1 + a_2 x_2 + ... + a_n x_n),$$

where $\mathbf{x} = (x_1, x_2, ..., x_n) \in R^n$ , $\mathbf{a} = (a_1, a_2, ..., a_n) \in R^n \setminus \{0\}$ is a fixed vector (direction) and $f : R \to R$ is a univariate function. These functions arise naturally in various fields. They arise in computerized tomography (see: [1, 2]), statistics (see: [3, 4]), data analysis (see: [5, 6]) and neural networks (see: [7, 8]). These functions are also used in modern approximation theory as an effective tool for approximating multivariate functions (see: [9, 10]). We refer the reader to the monographs of A.Pinkus [11] and V.Ismailov [12] for a detailed and systematic study of ridge functions.

One of the basic problems concerning the approximation by sums of ridge functions is the problem of the representability of a given multivariate function $F$ by sums of ridge functions. That is, assume we are given a multivariate function $F : R^n \to R$, and fixed pairwise linearly independent vectors $\mathbf{a}^k \in R^d$ , $k = \overline{1, m}$. It is required to find a condition under which the function $F$ can be represented in the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^k \cdot \mathbf{x}\right),$$

where $f_k : R \to R$ , $k = \overline{1, m}$ are the univariate functions. This problem has a simple solution if the dimension of the space is $n = 2$ and a given function $F(x, y)$ has partial derivatives up to $m$-th order. For the representation of the function $F(x, y)$ in the form

$$F(x, y) = \sum_{k=1}^{m} f_k (a_k x + b_k y)$$

it is necessary and sufficient that

$$\prod_{k=1}^{m} \left( b_k \frac{\partial}{\partial x} - a_k \frac{\partial}{\partial y} \right) F = 0 .$$

In the case of more than two variables, this condition changes slightly.

**Theorem 1** (P.Diaconis, M.Shahshahani [13]). Let $\mathbf{a}^1, ..., \mathbf{a}^m$ be pairwise linearly independent vectors in $R^n$ . Let $H^k$ denote the hyperplane

$\{\mathbf{c} \in R^n : \mathbf{c} \cdot \mathbf{a}^k = 0\}$, $k = \overline{1,m}$. Then a function $F \in C^{(m)}(R^n)$ can be represented in the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k(\mathbf{a}^k \cdot \mathbf{x}) + P_{m-1}(\mathbf{x}),$$

where $P_{m-1}(\mathbf{x})$ is a polynomial of degree less than $m$, if and only if

$$\prod_{k=1}^{m} \left( \sum_{s=1}^{n} c_s^{(k)} \frac{\partial}{\partial x_s} \right) f = 0$$

for all vectors $\mathbf{c}^k = (c_1^k, \dots, c_n^k) \in H^k$, $k = \overline{1,m}$.

**Remark 1.** There are examples showing that one cannot simply dispense with the polynomial $P_{m-1}(\mathbf{x})$ in the above theorem.

A multivariate function $F : R^n \to R$ of the form

$$F(\mathbf{x}) = f(\mathbf{a}^1 \cdot \mathbf{x}, \dots, \mathbf{a}^d \cdot \mathbf{x})$$

is called a generalized ridge function, where $\mathbf{a}^i \in R^n$, $i = \overline{1,d}$ are fixed linearly independent vectors (directions), $1 \le d < n$, and $f$ is a real-valued function defined on $R^d$. For $d = 1$, generalized ridge function reduces to a ridge function.

In this paper, we find a condition under which the smooth trivariate function $F(x_1, x_2, x_3)$ can be represented by sums of generalized ridge functions and give a partial solution of the problem, posted by A.Pinkus in his monograph "Ridge Functions".

## 2. Main results

**Definition 1.** Let $\{\mathbf{a}^1, \dots, \mathbf{a}^d\}$ and $\{\mathbf{b}^1, \dots, \mathbf{b}^d\}$, $1 \le d < n$, be linear independent vector systems in $R^n$. If

$$span\{\mathbf{a}^1, \dots, \mathbf{a}^d\} = span\{\mathbf{b}^1, \dots, \mathbf{b}^d\}$$

then the systems $\{\mathbf{a}^1, \dots, \mathbf{a}^d\}$ and $\{\mathbf{b}^1, \dots, \mathbf{b}^d\}$ are called eqiavalent, otherwise, that is, if

$$span\{\mathbf{a}^1,\ldots,\mathbf{a}^d\} \neq span\{\mathbf{b}^1,\ldots,\mathbf{b}^d\},$$

then the systems $\{\mathbf{a}^1,\ldots,\mathbf{a}^d\}$ and $\{\mathbf{b}^1,\ldots,\mathbf{b}^d\}$ are called non-eqiavalent.

**Remark 2.** Obviously, if the systems $\{\mathbf{a}^1,\ldots,\mathbf{a}^d\}$ and $\{\mathbf{b}^1,\ldots,\mathbf{b}^d\}$ are eqiavalent, then any generalized ridge function of the form $F(\mathbf{x}) = f(\mathbf{a}^1\cdot\mathbf{x},\ldots,\mathbf{a}^d\cdot\mathbf{x})$ also has the form $F(\mathbf{x}) = g(\mathbf{b}^1\cdot\mathbf{x},\ldots,\mathbf{b}^d\cdot\mathbf{x})$. Therefore, when defining a generalized ridge function, without loss of generality, we can assume that the vectors $\mathbf{a}^1,\ldots,\mathbf{a}^d$ are unit and mutually perpendicular.

Let's consider the following problem: assume we are given a multivariate function $F: R^n \to R$, and fixed pairwise non-eqiavalent vector systems $\{\mathbf{a}^{(1),1},\ldots,\mathbf{a}^{(1),d}\},\ldots,\{\mathbf{a}^{(m),1},\ldots,\mathbf{a}^{(m),d}\}$ in $R^n$. It is required to find a condition under which the function $F$ can be represented in the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^{(k),1}\cdot\mathbf{x},\ldots,\mathbf{a}^{(k),d}\cdot\mathbf{x}\right),$$

where $f_k : R^d \to R$, $k=\overline{1,m}$ are the real-valued functions.

In this paper we give a solution to this problem in the case of $n=3,\ d=2$.

**Theorem 2.** Let $\{\mathbf{a}^1,\mathbf{b}^1\},\ldots,\{\mathbf{a}^m,\mathbf{b}^m\}$ pairwise non-eqiavalent vector systems in $R^3$. Then the function $F \in C^{(m)}(R^3)$ can be represented in the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^k\cdot\mathbf{x},\mathbf{b}^k\cdot\mathbf{x}\right), \quad \mathbf{x}\in R^3, \tag{1}$$

if and only if

$$\frac{\partial^m F}{\partial l_1 .. \partial l_m}(\mathbf{x}) = 0, \quad \mathbf{x}\in R^3, \tag{2}$$

where $l_k \in R^3$ is a unit vector, perpendicular to the vectors $\mathbf{a}^k$ and $\mathbf{b}^k$, $k=\overline{1,m}$.

At first, we prove the auxiliary lemma.

**Lemma 1.** Let $\mathbf{a}$ and $\mathbf{b}$ any linearly independent vectors in $R^3$ and the vector $l \in R^3$ is not perpendicular to the vector space $span\{\mathbf{a},\mathbf{b}\}$. Then for any function $\varphi = \varphi(u,v) \in C^{(1)}(R^2)$ there exist a continuously differentiable generalized ridge function of the form $\Phi(\mathbf{x}) = f(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x})$ such that

$$\frac{\partial \Phi}{\partial l}(\mathbf{x}) = \varphi(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x}) \tag{3}$$

for any $\mathbf{x} \in R^3$.

**Proof of Lemma 1.** It follows from Remark 2 that without loss of generality, we can assume that the vectors $\mathbf{a}$ and $\mathbf{b}$ are unit and perpendicular. Denote by $\mathbf{c}$ the unit vector, perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$. Let

$$l = \alpha\cdot\mathbf{a} + \beta\cdot\mathbf{b} + \gamma\cdot\mathbf{c}.$$

As the vector $l \in R^3$ is not perpendicular to the vector space $span\{\mathbf{a},\mathbf{b}\}$, then

$$\alpha^2 + \beta^2 > 0 .$$

Denote

$$\Phi(\mathbf{x}) = \frac{1}{\alpha^2+\beta^2}\int\limits_0^{\alpha\cdot\mathbf{a}\cdot\mathbf{x}+\beta\cdot\mathbf{b}\cdot\mathbf{x}} \varphi\left(\frac{\alpha t + \beta^2\mathbf{a}\cdot\mathbf{x} - \alpha\beta\cdot\mathbf{b}\cdot\mathbf{x}}{\alpha^2+\beta^2}, \frac{\beta t - \alpha\beta\cdot\mathbf{a}\cdot\mathbf{x} + \alpha^2\mathbf{b}\cdot\mathbf{x}}{\alpha^2+\beta^2}\right)dt, \quad \mathbf{x}\in R^3 .$$

It follows from the equations

$$\frac{\partial \Phi}{\partial \mathbf{a}}(\mathbf{x}) = \frac{\alpha}{\alpha^2+\beta^2}\cdot\varphi(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x}) +$$

$$+\frac{1}{\alpha^2+\beta^2}\int\limits_0^{\alpha\cdot\mathbf{a}\cdot\mathbf{x}+\beta\cdot\mathbf{b}\cdot\mathbf{x}}\left(\frac{\beta^2}{\alpha^2+\beta^2}\cdot\frac{\partial\varphi}{\partial u}(s_1(t,\mathbf{x})) - \frac{\alpha\beta}{\alpha^2+\beta^2}\cdot\frac{\partial\varphi}{\partial v}(s_2(t,\mathbf{x}))\right)dt,$$

$$\frac{\partial \Phi}{\partial \mathbf{b}}(\mathbf{x}) = \frac{\beta}{\alpha^2+\beta^2}\cdot\varphi(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x}) +$$

$$+\frac{1}{\alpha^2+\beta^2}\int\limits_0^{\alpha\cdot\mathbf{a}\cdot\mathbf{x}+\beta\cdot\mathbf{b}\cdot\mathbf{x}}\left(-\frac{\alpha\beta}{\alpha^2+\beta^2}\cdot\frac{\partial\varphi}{\partial u}(s_1(t,\mathbf{x})) + \frac{\alpha^2}{\alpha^2+\beta^2}\cdot\frac{\partial\varphi}{\partial v}(s_2(t,\mathbf{x}))\right)dt,$$

$$\frac{\partial \Phi}{\partial \mathbf{c}}(\mathbf{x}) = 0,$$

where

$$s_1(t,\mathbf{x}) = \frac{\alpha t + \beta^2 \mathbf{a}\cdot\mathbf{x} - \alpha\beta\cdot\mathbf{b}\cdot\mathbf{x}}{\alpha^2+\beta^2}, \quad s_2(t,\mathbf{x}) = \frac{\beta t - \alpha\beta\cdot\mathbf{a}\cdot\mathbf{x} + \alpha^2\mathbf{b}\cdot\mathbf{x}}{\alpha^2+\beta^2},$$

that

$$\frac{\partial\Phi}{\partial l}(\mathbf{x}) = \alpha\frac{\partial\Phi}{\partial\mathbf{a}}(\mathbf{x}) + \beta\frac{\partial\Phi}{\partial\mathbf{b}}(\mathbf{x}) + \gamma\frac{\partial\Phi}{\partial\mathbf{c}}(\mathbf{x}) = \varphi(\mathbf{a}\cdot\mathbf{x},\mathbf{b}\cdot\mathbf{x}).$$

On the other side, it follows from $\frac{\partial\Phi}{\partial\mathbf{c}}(\mathbf{x}) = 0$ that the function $\Phi$ is of the form $\Phi(\mathbf{x}) = f(\mathbf{a}\cdot\mathbf{x},\mathbf{b}\cdot\mathbf{x})$. This completes the proof of the lemma.

**Proof of Theorem 2. Necessity.** Let the function $F \in C^{(m)}(R^3)$ is of the form (1). For any $\mathbf{x}\in R^3$ and $\mathbf{h}\in R^3$ we denote by $\Delta_{\mathbf{h}}F(\mathbf{x})$ the increment

$$\Delta_{\mathbf{h}}F(\mathbf{x}) = F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x})$$

of a function $F$. Then it follows from (1) that for any $\mathbf{x}\in R^3$ and for any $t_1,\dots,t_m \in R$

$$\Delta_{t_1\cdot l_1}\Delta_{t_2\cdot l_2}..\Delta_{t_m\cdot l_m}F(\mathbf{x}) = 0, \tag{4}$$

where $l_k \in R^3$ is a unit vector, perpendicular to the vectors $\mathbf{a}^k$ and $\mathbf{b}^k$, $k=\overline{1,m}$. It follows from (4) that for any $\mathbf{x}\in R^3$

$$\frac{\partial^m F}{\partial l_1..\partial l_m}(\mathbf{x}) = \lim_{t_1\to 0,\dots,t_m\to 0}\frac{\Delta_{t_1\cdot l_1}\Delta_{t_2\cdot l_2}..\Delta_{t_m\cdot l_m}F(\mathbf{x})}{t_1\cdot t_2\cdot\ldots\cdot t_m} = 0.$$

**Sufficiency.** Let the function $F\in C^{(m)}(R^3)$ satisfy condition (2) for any $\mathbf{x}\in R^3$. Let us write equation (2) in the form

$$\frac{\partial}{\partial\partial_1}\left[\frac{\partial^{m-1}F}{\partial l_2..\partial l_m}\right](\mathbf{x}) = 0. \tag{5}$$

It follows from (5) that the partial derivative $\frac{\partial^{m-1}F}{\partial l_2..\partial l_m}$ of the function $F$ independent of the direction $l_1$. Therefore there exist a function $\varphi_1 : R^2 \to R$ such that

$$\frac{\partial^{m-1}F}{\partial l_2..\partial l_m}(\mathbf{x})=\varphi_1\left(\mathbf{a}^1\cdot\mathbf{x},\mathbf{b}^1\cdot\mathbf{x}\right),\ \ \mathbf{x}\in R^3. \tag{6}$$

From condition $F\in C^{(m)}\left(R^3\right)$ we obtain that $\varphi_1\in C^{(1)}\left(R^2\right)$. Now let us write equation (6) in the form

$$\frac{\partial}{\partial\partial_2}\left[\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}\right](\mathbf{x})=\varphi_1\left(\mathbf{a}^1\cdot\mathbf{x},\mathbf{b}^1\cdot\mathbf{x}\right),\ \ \mathbf{x}\in R^3. \tag{7}$$

It follows from lemma 1 that there exists a continuously differentiable generalized ridge function of the form

$$\Phi_1(\mathbf{x})=g_1\left(\mathbf{a}^1\cdot\mathbf{x},\mathbf{b}^1\cdot\mathbf{x}\right) \tag{8}$$

such that

$$\frac{\partial\Phi_1}{\partial l_2}(\mathbf{x})=\varphi_1\left(\mathbf{a}^1\cdot\mathbf{x},\mathbf{b}^1\cdot\mathbf{x}\right),\ \ \mathbf{x}\in R^3. \tag{9}$$

It follows from (7) and (9) that for any $\mathbf{x}\in R^3$

$$\frac{\partial}{\partial\partial_2}\left[\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}-\Phi_1\right](\mathbf{x})=0.$$

Then the function $\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}-\Phi_1$ independent of the direction $l_2$. Therefore there exist a function $\varphi_2:R^2\to R$ such that

$$\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}(\mathbf{x})-\Phi_1(\mathbf{x})=\varphi_2\left(\mathbf{a}^2\cdot\mathbf{x},\mathbf{b}^2\cdot\mathbf{x}\right),\ \ \mathbf{x}\in R^3. \tag{10}$$

Since the functions $\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}$ and $\Phi_1$ are continuously differentiable, then we get that the function $\varphi_2$ also continuously differentiable in $R^2$. It follows from (8) and (10) that

$$\frac{\partial^{m-2}F}{\partial l_3..\partial l_m}(\mathbf{x})=g_1\left(\mathbf{a}^1\cdot\mathbf{x},\mathbf{b}^1\cdot\mathbf{x}\right)+\varphi_2\left(\mathbf{a}^2\cdot\mathbf{x},\mathbf{b}^2\cdot\mathbf{x}\right),\ \ \mathbf{x}\in R^3.$$

Continuing the above process, until it reaches the function $F$, we obtain the desired result. This completes the proof of the theorem.

## 3. The smoothness problem in generalized ridge function representation

Assume we are given a function $F : R^n \to R$ of the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^{(k),1} \cdot \mathbf{x}, \ldots, \mathbf{a}^{(k),d} \cdot \mathbf{x}\right), \tag{11}$$

where $\left\{\mathbf{a}^{(1),1}, \ldots, \mathbf{a}^{(1),d}\right\}, \ldots, \left\{\mathbf{a}^{(m),1}, \ldots, \mathbf{a}^{(m),d}\right\}$ are fixed pairwise non-eqiavalent vector systems in $R^n$ and $f_k : R^d \to R$, $k = \overline{1,m}$ are the real-valued functions. Assume, in addition, that $F$ is of a certain smoothness class, that is, $F \in C^{(s)}\left(R^n\right)$, where $s \geq 0$ (with the convention that $C^{(0)}\left(R^n\right) = C\left(R^n\right)$). What can we say about the smoothness of the $f_k$ ? Do the $f_k$ necessarily inherit all the smoothness properties of the $F$ ?

If $d = 1$ and $m = 1$ or $m = 2$ the answer is yes (see [12]). If $d = 1$ and $m \geq 3$ the picture drastically changes. For $m = 3$, there are smooth functions which decompose into sums of very badly behaved ridge functions. For example, if $h_1$ be any non linear solution of the Cauchy Functional Equation

$$h(x + y) = h(x) + h(y),$$

then the zero function can be represented as

$$0 = h_1(x) + h_1(y) - h(x + y). \tag{12}$$

Note that the functions involved in (12) are bivariate ridge functions with the directions (1,0), (0,1) and (1,1), respectively. This example shows that for smoothness of the representation (11) one must impose additional conditions on the representing functions $f_k$, $k = \overline{1,m}$.

In case $d = 1$ it was first proved by M.Buhmann and A.Pinkus [14] that if in (11) $F \in C^{(s)}\left(R^n\right)$, $s \geq m-1$ and $f_k \in L^1_{loc}(R)$ for each $k$, then $f_k \in C^{(s)}(R)$ for $k = \overline{1,m}$. Later, A.Pinkus [15] generalized extensively this result. He solved this problem for any $s \in Z_+$, while imposing weaker conditions on the functions $f_k$.

In case $d \geq 2$ the situation is slightly more problematic. Consider, for example, the case $d=2$, $n=3$, $m=2$, $\mathbf{a}^{(1),1}=(1,0,0)$, $\mathbf{a}^{(1),2}=(0,1,0)$, $\mathbf{a}^{(2),1}=(0,1,0)$, $\mathbf{a}^{(2),2}=(0,0,1)$. Thus

$$F(x_1,x_2,x_3)= f_1(x_1,x_2)+ f_2(x_2,x_3).$$

Setting $f_1(x_1,x_2)=g(x_2)$ and $f_2(x_2,x_3)=-g(x_2)$ for any arbitrary univariate function $g$, we have

$$0= f_1(x_1,x_2)+ f_2(x_2,x_3),$$

and yet $f_1$ and $f_2$ do not exhibit any of the smoothness properties of the left-hand side of this equation.

Now consider the following natural and interesting question. Assume we are given a function $F \in C^{(s)}(R^n)$ of the form (11). Is it true that there will always exist $g_k \in C^{(s)}(R^d)$, $k=\overline{1,m}$ such that

$$F(\mathbf{x})= \sum_{k=1}^{m} g_k\left(\mathbf{a}^{(k),1}\cdot\mathbf{x},\ldots,\mathbf{a}^{(k),d}\cdot\mathbf{x}\right)?$$

This question was posed in M.Buhmann and A.Pinkus [14] for ridge function representation and Pinkus [11] for generalized ridge function representation. In [17, 18, 19], the authors gave a partial solution to the above representation problem for ridge function representation. In [19], this problem for ridge function representation is solved up to a multivariate polynomial:

**Theorem 3** (R.Aliev, V.İsmailov [19]). Assume a function $F \in C(R^n)$ is of the form

$$F(\mathbf{x})= \sum_{k=1}^{m} f_k\left(\mathbf{a}^k\cdot\mathbf{x}\right), \tag{13}$$

where $\mathbf{a}^1,\ldots,\mathbf{a}^m$ are pairwise linearly independent directions in $R^d$, $f_1,\ldots,f_m$ are arbitrarily behaved univariate functions. Then there exist continuous functions $g_k : R \to R$, $k=\overline{1,m}$, and a polynomial $P(x)$ of degree at most $m-1$ such that

$$F(\mathbf{x}) = \sum_{k=1}^{m} g_k\left(\mathbf{a}^k \cdot \mathbf{x}\right) + P(\mathbf{x}). \quad (14)$$

**Corollary 1** (R.Aliev, V.İsmailov [19]). Assume a function $F \in C^{(s)}\left(R^n\right)$ is of the form (13). Then there exist functions $g_k \in C^{(s)}(R)$, $k = \overline{1,m}$, and a polynomial $P(x)$ of degree at most $m-1$ such that (14) holds.

**Corollary 2** (R.Aliev, V.İsmailov [19]). Assume a function $F \in C^{(s)}\left(R^2\right)$ is of the form (13). Then there exist functions $g_k \in C^{(s)}(R)$, $k = \overline{1,m}$ such that

$$F(\mathbf{x}) = \sum_{k=1}^{m} g_k\left(\mathbf{a}^k \cdot \mathbf{x}\right).$$

In this section we give a partial solution posted problem for generalized ridge function representation.

**Theorem 4.** Assume a function $F \in C^{(m)}\left(R^3\right)$ is of the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^k \cdot \mathbf{x}, \mathbf{b}^k \cdot \mathbf{x}\right), \quad \mathbf{x} \in R^3. \quad (15)$$

Then there exist functions $g_k \in C^{(1)}\left(R^2\right)$, $k = \overline{1,m}$, such that

$$F(\mathbf{x}) = \sum_{k=1}^{m} g_k\left(\mathbf{a}^k \cdot \mathbf{x}, \mathbf{b}^k \cdot \mathbf{x}\right), \quad \mathbf{x} \in R^3. \quad (16)$$

**Proof of theorem 4.** Let the function $F \in C^{(m)}\left(R^3\right)$ is of the form (15). Then it follows from Theorem 2 that

$$\frac{\partial^m F}{\partial l_1 .. \partial l_m}(\mathbf{x}) = 0$$

for any $\mathbf{x} \in R^3$, where $l_k \in R^3$ is a unit vector, perpendicular to the vectors $\mathbf{a}^k$ and $\mathbf{b}^k$, $k = \overline{1,m}$. Then from the proof of the sufficiency of Theorem 2 it is clear that there exist continuously differentiable functions $g_k : R^2 \to R$, $k = \overline{1,m}$, such that (16) is satisfied. This completes the proof of the theorem.

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