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On the direct correspondence between the trigonometric Pöschl-Teller potential well and the quantum singular oscillator with the position-dependent mass

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Abstract

We show that the trigonometric Pöschl-Teller potential well problem of the non-relativistic quantum mechanics is equivalent to a certain model of a linear singular oscillator with the position-dependent mass of the form $M(x) = a^2 m_0/(a^2 - x^2)$, $0 \leq x \leq a$. We found an explicit form of the functions and energy of the wave functions and discrete energy spectrum for this model. Wave functions are expressed through the Jacobi polynomials. At the limit when $a \to \infty$ equation of the motion, wave functions and energy spectrum of the model correctly reduce to corresponding results of the usual non-relativistic linear singular oscillator with a constant mass m_0.

Keywords: *Schrödinger equation, quantum singular oscillator, trigonometric Pöschl-Teller potential, position-dependent mass*
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1. Introduction

The position-dependent mass concept is one of the successfully developed directions of modern quantum mechanics during the last decades [1-8]. Their main attractivity is due to two important reasons: the first reason is that they have a huge number of applications for an explanation of the electronic structures of the various solid-state heterostructures thanks to the possibility of introducing the confinement effect successfully through the position-dependent mass; the second reason

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is related with the certain analytical expressions of the mass varying with position, which leads to the exact solutions of the corresponding Schrödinger equation. All obtained analytical solutions of the Schrödinger equation can reproduce correctly the known similar quantum mechanical systems if one manages to remove the confinement effect and return the position-dependent mass to its constant or homogeneous analogue.

Recently, we have developed the quantum singular oscillator model with the position-dependent mass [9]. It is well known that the singular oscillator has exact solutions in the framework of non-relativistic quantum mechanics [10]. Its wavefunctions of the stationary states are expressed through the generalized Laguerre polynomials and the energy spectrum exhibits close similarity with the non-relativistic quantum harmonic oscillator – it is equidistant and consists of an infinite number of energy levels. At the same time, the trigonometric Pöschl-Teller potential well is also one of the exactly solvable problems of the one-dimensional non-relativistic quantum mechanics [11, 12]. It has an energy spectrum with drastically different behavior than the non-relativistic quantum harmonic oscillator energy spectrum, additionally, its wavefunctions of the stationary states are expressed through the Jacobi polynomials. In this paper, we are going to show that the quantum singular oscillator with certain analytical position dependencies on its mass and the trigonometric Pöschl-Teller potential well problem of the non-relativistic quantum mechanics can be directly connected through the elegant mathematical transform tool between both of their exact solutions.

2. One-dimensional Pöschl-Teller potential well

One writes the following Schrödinger equation for the one-dimensional Pöschl-Teller potential well [11, 12]:

$$\left\{-\frac{\hbar^2}{2m_0}\frac{d^2}{dx^2}+\frac{1}{2}V_0\left[\frac{\kappa(\kappa-1)}{sin^2\,\alpha x}+\frac{\lambda(\lambda-1)}{cos^2\,\alpha x}\right]\right\}\psi=E\psi. \tag{2.1}$$

Here, $\kappa>1, \lambda>1$ and the notation $V_0=\hbar^2\alpha^2/m_0$ is introduced for simplicity. In general, the behavior of the one-dimensional Pöschl-Teller potential well is periodical. Therefore, we are going to study only the region $0\le x\le\pi/2\alpha$, where the potential becomes infinitely high at the borders of this region. It means that the wavefunction has to possess the following boundary condition:

$$\psi(0)=\psi(\pi/2\alpha)=0. \tag{2.2}$$

Introduction of the new variable

$$y=sin^2\,\alpha x\,,\qquad 0\le y\le 1 \tag{2.3}$$

converts eq. (2.1) as follows:

$$y(1-y)u'' + \left(\tfrac{1}{2} - y\right)u' + \tfrac{1}{4}\left[\frac{k^2}{\alpha^2} - \frac{\kappa(\kappa-1)}{y} + \frac{\lambda(\lambda-1)}{1-y}\right]u = 0 \qquad (2.4)$$

where, the notations $k = \sqrt{2m_0 E/\hbar}$ and $u(y) = \psi(x)$ are introduced now.
As it is shown in [1], the solution to eq. (2.4) in terms of the stationary wavefunctions can be written as a ${}_2F_1$ hypergeometric function of the following behavior:

$$u_n(x) = c_n' \, sin^{\kappa}\, \alpha x \, cos^{\lambda}\, \alpha x \;\; {}_2F_1\left(-n, \kappa+\lambda+n; \kappa+\tfrac{1}{2}| \, sin^2\, \alpha x\right). \qquad (2.5)$$

Exact expression of the energy spectrum corresponding to the analytical solution (2.5) will be as

$$E_n = \frac{1}{2}V_0(\kappa+\lambda+2n)^2, \qquad n = 0, 1, 2, 3, \cdots \qquad (2.6)$$

Now, one can use the following definition of the Jacobi polynomials thwough the ${}_2F_1$ hypergeometric functions of the (2.5) type as follows [13]:

$$P_n^{(\mu,\nu)}(x) = \frac{(\mu+1)_n}{n!} \;\; {}_2F_1\left(-n, \kappa+\mu+\nu+1; \mu+1| \frac{1-z}{2}\right), \qquad (2.7)$$

which allows us to rewrite eq.(2.5) in terms of the Jacobi polynomials:

$$\psi_n(x) = c_n \, sin^{\kappa}\, \alpha x \, cos^{\lambda}\, \alpha x \, P_n^{\left(\kappa-\frac{1}{2}, \lambda-\frac{1}{2}\right)}(cos\, 2\alpha x). \qquad (2.8)$$

c_n is the parameter of the orthonormalization and it can be computed analytically from the following orthonormalization integral:

$$\int_0^{\pi/2\alpha} \psi_n{}^*(x)\psi_n(x)dx = 1. \qquad (2.9)$$

3. The singular oscillator model with the position-dependent mass

In this section, we will construct the linear singular oscillator model. We assume that its mass varies with position. Another initial assumption is that the model under construction is located in the infinitely deep potential well with the width a. In order to apply these two initial conditions, one introduces the following potential:

$$V(x) = \begin{cases} \frac{1}{2}M(x)\omega^2 x^2 + \dfrac{g}{x^2}, & 0 \le x \le a \\ \infty, & x < 0 \; and \; x > a \end{cases} \qquad (3.1)$$

Next, we need to choose the free Hamiltonian $\widehat{H}_0$ and the function of the mass changing with position $M \equiv M(x)$. Definition of the free Hamiltonian initially requires that it has to preserve its hermiticity under the any generalization. Then, one supposes for it:

$$\hat{H}_0 = \hat{H}_0^{\ BD} + V_{free}(x). \tag{3.2}$$

Here, $\hat{H}_0^{\ BD}$ is the BenDaniel-Duke approach to the kinetic energy operator for the case of the position-dependent mass generalization and it has the following exact expression:

$$\hat{H}_0^{\ BD} = -\frac{\hbar^2}{2}\frac{d}{dx}M^{-1}\frac{d}{dx}, \tag{3.3}$$

and $V_{free}(x)$ can be defined as follows [8]:

$$V_{free}(x) = A_f\frac{\hbar^2 M'}{2M^2} - B_f\frac{\hbar^2 M''}{2M^3}. \tag{3.3'}$$

Here, A_f and B_f are the arbitrary real constants.

We select the mass changing with position as follows:

$$M(x) = a^2 m_0/(a^2 - x^2), \qquad 0 \le x \le a \tag{3.4}$$

Then, one performs simple computations and obtains that

$$\frac{M'}{M} = \frac{2x}{a^2 - x^2}, \quad \frac{M''}{M} = \frac{2}{a^2 - x^2} + \frac{8x^2}{(a^2 - x^2)^2}. \tag{3.5}$$

Its substitution at (3.3) yields the following analytical expression for the free potential $V_{free}(x)$:

$$V_{free}(x) = \frac{1}{a^2 m_0}\left[-\frac{1}{2}\hbar^2 B_f + \frac{2\hbar^2\left(A_f - B_f\right)x^2}{a^2 - x^2}\right]. \tag{3.6}$$

Then, our full Hamiltonian should be defined as follows:

$$\hat{H} = \hat{H}_0 + V(x) = \hat{H}_0^{\ BD} + V_{eff}(x) \tag{3.7}$$

where,

$$V_{eff}(x) = V_{free}(x) + V(x). \tag{3.8}$$

Their substitution at the one-dimensional Schrödinger equation $\hat{H}\psi = E\psi$ yields

$$(a^2 - x^2)\frac{d^2\psi}{dx^2} - 2x\frac{d\psi}{dx} + \left(\frac{2m_0 a^2 E}{\hbar^2} - \frac{2m_0 a^2 g}{\hbar^2 x^2} - \frac{\lambda_0^{\ 4} a^4 x^2}{a^2 - x^2}\right)\psi = 0,$$

$$\lambda_0 = \sqrt{m_0\omega/\hbar}. \tag{3.9}$$

Introduction of the dimensionless variable $\xi = x/a$ simplifies (3.9) as follows:

$$(1-\xi^2)\psi'' - 2\xi\psi' + \left(c_0 - \frac{c_1}{\xi^2} - \frac{c_2\xi^2}{1-\xi^2}\right)\psi = 0, \qquad 0 \le \xi \le 1, \qquad (3.10)$$

Here, $\psi' \equiv d\psi/d\xi$, $\psi'' \equiv d^2\psi/d\xi^2$ as well as the following constants are introduced for simplicity of the second order differential equation:

$$c_0 = \frac{2m_0a^2E}{\hbar^2}, \qquad c_1 = \frac{2m_0g}{\hbar^2}, \qquad c_2 = {\lambda_0}^4a^4. \qquad (3.11)$$

More simplification of the above equation requires introduction of the one more new dimensionless variable $\xi = \sin\alpha z$, where now the condition $0 \le z \le \pi/2\alpha$ holds. Then, one obtains that

$$\frac{d}{d\xi} = \frac{1}{\alpha\cos\alpha z}\frac{d}{dz}, \qquad \frac{d^2}{d\xi^2} = \frac{1}{\alpha^2\cos^2\alpha z}\left(\frac{d^2}{dz^2} + \alpha\tan\alpha z\frac{d}{dz}\right). \qquad (3.12)$$

Substitution of (3.12) as eq.(3.10) yields:

$$\left[\frac{d^2}{dz^2} - \alpha\tan\alpha z\frac{d}{dz} + \alpha^2\left(c_0 + c_2 - \frac{c_1}{\sin^2\alpha z} - \frac{c_2}{\cos^2\alpha z}\right)\right]\psi = 0. \qquad (3.13)$$

Let's try to rewrite eq. (3.13) in the same manner as eq. (2.1). One needs to note that eq. (3.13) is a second-order differential equation of the following general form:

$$a_2\psi'' + a_1\psi' + a_0\psi = 0, \qquad (3.14)$$

where,

$$a_0 = c_0 + c_2 - \frac{c_1}{\sin^2\alpha z} - \frac{c_2}{\cos^2\alpha z}, \qquad a_1 = -\alpha\tan\alpha z, \qquad a_2 = 1 \qquad (3.15)$$

Then, we look for the solution of (3.14) as

$$\psi = f\cdot\varphi. \qquad (3.16)$$

Substitution of (3.16) at eq. (3.14) leads to the following equation for φ:

$$b_2\varphi'' + b_1\varphi' + b_0\varphi = 0, \qquad (3.17)$$

where,

$$b_0 = a_0 + a_1\frac{f'}{f} + a_2\frac{f''}{f}, \qquad b_1 = a_1 + 2a_2\frac{f'}{f}, \qquad b_2 = a_2 = 1. \qquad (3.18)$$

Comparison of (3.17) and (2.1) requires that $b_1 = 0$ should hold. This condition leads to the following analytical expression for f:

$$f = 1/\sqrt{\cos\alpha z}. \qquad (3.19)$$

Hence, one easily computes that

$$\frac{f'}{f} = \frac{1}{2}\alpha \tan \alpha z\,, \qquad \frac{f''}{f} = \frac{\alpha^2}{4} - \frac{3\alpha^2}{4\cos^2 \alpha z} \tag{3.20}$$

Their substitution at (3.18) leads to the following analytical expression for b_0:

$$b_0 = \alpha^2\left(c_0 + c_2 + \frac{1}{4}\right) - \alpha^2\left(\frac{c_1}{\sin^2 \alpha z} + \frac{\bar{c}_2}{\cos^2 \alpha z}\right), \quad \bar{c}_2 = c_2 - \frac{1}{4}. \tag{3.21}$$

Now, one observes that eq.(3.14) becomes as follows:

$$\left[\frac{d^2}{dz^2} + \alpha^2\left(c_0 + c_2 + \frac{1}{4}\right) - \alpha^2\left(\frac{c_1}{\sin^2 \alpha z} + \frac{\bar{c}_2}{\cos^2 \alpha z}\right)\right]\psi = 0, \tag{3.22}$$

and it almost completely overlaps with eq. (2.1) that is the Schrödinger equation of the trigonometric Pöschl-Teller potential well problem. In order to show complete overlap, one needs to multiply both sides of eq. (2.1) to $-2m_0/\hbar^2$. Such a multplication gives

$$\left\{\frac{d^2}{dx^2} - \frac{m_0 V_0}{\hbar^2}\left[\frac{\kappa(\kappa-1)}{\sin^2 \alpha x} + \frac{\lambda(\lambda-1)}{\cos^2 \alpha x}\right] + k^2\right\}\psi = 0 \tag{3.23}$$

Performing simple computations, one observes that

$$m_0 V_0/\hbar^2 = \alpha^2.$$

Therefore, one obtains slight change in eq.(3.23) as follows:

$$\left\{\frac{d^2}{dx^2} - \alpha^2\left[\frac{\kappa(\kappa-1)}{\sin^2 \alpha x} + \frac{\lambda(\lambda-1)}{\cos^2 \alpha x}\right] + k^2\right\}\psi = 0. \tag{3.24}$$

Now, comparison of (3.24) with (3.22) allows to obtain exact expression of the energy spectrum

$$E_n^{MSO} = \frac{\hbar^2\alpha^2}{2m_0}\left(2n + {\lambda_0}^2 a^2 + \frac{1}{2}\sqrt{1 + 8m_0 g/\hbar^2} + 1\right)^2,$$

$$c_1 = \kappa(\kappa-1), \bar{c}_2 = \lambda(\lambda-1). \tag{3.25}$$

The wavefunctions of the stationary states of both quantum models completely overlap.

4. Conclusions

The trigonometric Pöschl-Teller potential well problem is one of the thoroughly studied quantum mechanics problems. However, the quantum singular oscillator with certain analytical position dependencies on its mass is one of the recently developed concepts of the non-relativistic quantum mechanics. These two problems are exactly solvable and stationary wavefunctions of both these problems are written down in terms of the Jacobi polynomials. We have shown that the quantum singular oscillator with certain analytical position dependencies on its mass and the trigonometric Pöschl-Teller potential well problem of the non-relativistic quantum mechanics can be directly connected through the elegant mathematical transform tool between both of their exact solutions.

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