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# NODAL SOLUTIONS OF SOME NONLINEAR PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER

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**Abstract**

In this paper we consider the certain nonlinear boundary value problem for ordinary differential equations of fourth order which contain some parameter. The values of this parameter are determined at which nodal solutions to the problem under consideration exist.

***Keywords:*** *nonlinear problem, nodal zero, global bifurcation*

***Mathematics Subject Classification*** *(2020): 34A30, 34B15, 34C23, , 47J10, 47J15*

## 1. Introduction

Consider the following nonlinear problem

$$(p(x)y''(x))'' - (q(x)y'(x))' = \omega r(x) f(y(x)), \quad 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2}$$

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$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{4}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{5}$$

where $\lambda$ is a real parameter, $Ty \equiv (py'')' - qy'$, the functions $p(x)$ and $r(x)$ are positive on $[0,l]$, $q(x)$ is a nonnegative on $[0,l]$, $p' \in AC[0,l]$, $q \in AC[0,l]$, $r \in C[0,l]$, $\omega \in R$ and $\alpha, \beta, \gamma, \delta \in [0, \pi/2]$. Moreover, the real-valued function $f \in C(R)$ and there exist positive constants $f_0$ and $f_\infty$ such that

$$\lim_{|y| \to 0} \frac{f(y)}{y} = f_0 \text{ and } \lim_{|y| \to +\infty} \frac{f(y)}{y} = f_\infty . \tag{6}$$

Nonlinear boundary value problems for ordinary differential equations play an important role in modern mathematics, since they describe various processes in physics, mechanics, biology and other areas of natural science (see, for example, [3-5, 13]). Note that problem (1.1)-(1.5) arises when studying the bending of an inhomogeneous Euler-Bernoulli beam, in the cross sections of which a longitudinal force acts, at the boundary points of which various conditions are imposed.

The study of nodal solutions of nonlinear Sturm-Liouville problems and nonlinear problems for ordinary differential equations of fourth order has been the subject of many papers (see, for example, [1-3, 6-12, 14, 15] and their bibliography). It should be noted that in these works nonlinear boundary value problems of the fourth order were considered only in special cases and the existence of positive and negative solutions was established in them.

In this paper we consider a more general case and prove the existence of solutions having any number of simple zeros in the interval.

## 2. Preliminary

By $(b.c.)$ we denote the set of boundary conditions (2)-(5). Consider the linear problem

$$\begin{cases} (p(x)y''(x))'' - (q(x)y'(x))' = \lambda\tau(x)y(x), \ 0 < x < l, \\ y \in (b.c.). \end{cases} \tag{7}$$

Problem (7) was considered in the paper [3], where it was shown that the eigenvalues of this problem are nonnegative, simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, for each $k \in \mathrm{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue $\lambda_k$ has exactly $k-1$ simple nodal zeros in the interval $(0, l)$. Note that the eigenfunctions of problem (7) also have other important properties that are possessed by functions from the classes $S_k^{\nu} \subset E$ constructed in paper [1], where $E$ is a Banach space $C^3[0,l] \cap (b.c.)$ with the norm $\| y \|_3 = \sum_{i=0}^{3} \| y^{(i)} \|_{\infty}, \| y \| = \max_{x \in [0,l\}} | y(x) |$.

Alongside the spectral problem we shall consider the following nonlinear eigenvalue problem

$$\begin{cases} (p(x)y''(x))'' - (q(x)y'(x))' = \lambda \tau(x) y(x) + g(x, y, y', y'', y''', \lambda), \ 0 < x < l, \\ y \in (b.c.), \end{cases} \tag{8}$$

where the nonlinear term $g$ is a real-valued continuous function on $[0,l] \times R^5$ and satisfies the following condition: for every bounded interval $\Lambda$,

$$g(x, y, s, v, w, \lambda) = o(| y | + | s | + | v | + | w |) \text{ as } | y | + | s | + | v | + | w | \to 0, \tag{9}$$

or

$$g(x, y, s, v, w, \lambda) = o(| y | + | s | + | v | + | w |) \text{ as } | y | + | s | + | v | + | w | \to +\infty, \tag{10}$$

uniformly in $(x, \lambda) \in [0, l] \times \Lambda$.

Global bifurcation of nontrivial solutions of problem (8) was considered in [1] in the case when condition (9) is satisfied, in [2] in the case when condition (10) is satisfied. Note that when condition (9) is satisfied, then the bifurcation from the line of trivial solutions is studied; when condition (10) is satisfied, then the bifurcation from the line $R \times \{\infty\}$ is studied.

According to [1, Theorem 1.1] and [2, Theorem 3.1], we have the following global bifurcation results for problem (8) under conditions (9) and (10), respectively.

**Theorem A.** *Let condition (9) be satisfied. Then for each* $k \in \mathrm{N}$ *and* *each* $\nu \in \{+,-\}$ *there exists a continuum* $C_k^\nu$ *of nontrivial solutions of problem (8) which meets* $(\lambda_k, 0)$, *lies in* $R \times S_k^\nu$ *and is unbounded in* $R \times E$.

**Theorem B.** *Let condition (10) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+,-\}$ *there exists a continuum* $D_k^\nu$ *of nontrivial solutions of problem (8) which meets* $(\lambda_k, \infty)$ *and has the following properties: (i) there exists a neighbourhood* $Q_k$ *of* $(\lambda_k, \infty)$ *in* $R \times E$ *such that* $D_k^\nu \setminus Q_k^\nu \subset R \times S_k^\nu$; *(ii) either* $D_k^\nu$ *meets* $D_{k'}^{\nu'}$ *through* $R \times S_{k'}^{\nu'}$ *for some* $(k', \nu') \neq (k, \nu)$, *or* $D_k^\nu$ *meets* $(\lambda, 0)$ *for some* $\lambda \in R$, *or the projection of* $D_k^\nu$ *onto* $R \times \{0\}$ *is unbounded.*

**Remark 1.** If condition (9) holds, then by theorem A the set $C_k^\nu$ is unbounded in $R \times E$, and consequently, either $C_k^\nu$ meets $(\lambda, \infty)$ for some $\lambda \in R$, or the projection of the set $C_k^\nu$ onto $R \times \{0\}$ is unbounded.

## 3. Existence of nodal solutions to problem (1)-(5)

This section is devoted to finding the interval of the parameter $\omega$, in which there are nodal solutions to problem (1)-(5), or more precisely, there are solutions contained in the classes $S_k^\nu, k \in \mathrm{N}, \nu \in \{+,-\}$.

**Lemma 1.** *The following relations hold:*

$$g_0(y) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to 0, \tag{11}$$

and

$$g_\infty(y) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to +\infty. \tag{12}$$

**Proof.** By (6) for the function $f$ we have the following representations

$$f(y) = f_0 y + g_0(y) \text{ and } f(y) = f_\infty y + g_\infty(y), \tag{13}$$

where

$$\lim_{|y| \to 0} \frac{g_0(y)}{y} = 0 \text{ and } \lim_{|y| \to +\infty} \frac{g_\infty(y)}{y} = 0. \tag{14}$$

It follows from (14) that for any sufficiently small $\varepsilon > 0$ there exist a sufficiently small $\sigma_\varepsilon > 0$ and a sufficiently large $\Delta_\varepsilon > 0$ such that

$$\frac{|g_0(y)|}{|y|} < \varepsilon \text{ for any } y \in R, 0 < |y| < \sigma_\varepsilon, \tag{15}$$

and

$$\frac{|g_\infty(y)|}{|y|} < \varepsilon \text{ for any } y \in R, |y| > \Delta_\varepsilon. \tag{16}$$

Then by (15) we have

$$\frac{|g_0(y)|}{|y|+|s|+|v|+|w|} < \varepsilon \text{ for any } y \in R, |y|+|s| < \sigma_\varepsilon. \tag{17}$$

In view of (13) we get

$$g_\infty(y) = f_0 y - f_\infty y + g_0(y), \; y \in R. \tag{18}$$

By condition $g_\infty(y) \in C(R)$ there exits positive constant $\kappa_\varepsilon$ such that

$$|g_\infty(y)| \le \kappa_\varepsilon \text{ for any } y \in R, \; |y| \le \Delta_\varepsilon. \tag{19}$$

Let $\Delta_{1\varepsilon} > \Delta_\varepsilon$ is chosen so that the inequality

$$\Delta_{1\varepsilon} > \frac{\kappa_\varepsilon}{\varepsilon} \tag{20}$$

holds.

Now let $|y|+|s|+|v|+|w| > \Delta_{1\varepsilon}$. Then by (16), (19) and (20) we obtain

$$\frac{|g_\infty(y)|}{|y|+|s|+|v|+|w|} < \frac{|g_\infty(y)|}{|y|} < \varepsilon \text{ for any } y \in R, |y| > \Delta_\varepsilon, \tag{21$_1$}$$

$$\frac{|g_\infty(y)|}{|y|+|s|+|v|+|w|} < \frac{\kappa_\varepsilon}{\Delta_{1\varepsilon}} < \varepsilon \text{ for any } y \in R, |y| \le \Delta_\varepsilon. \tag{21$_2$}$$

Thus, relations (11) and (12) follow directly from (17) and (21), respectively. The proof of this lemma is complete.

Consider the following eigenvalue problem

$$\begin{cases} (p(x)y''(x))'' - (q(x)y'(x))' = \lambda\omega\tau(x) f_0 y(x) + \tau(x)\omega \; g_0(y(x)), \; 0 < x < l, \\ y \in (b.c.). \end{cases} \tag{22}$$

It is obvious that the eigenvalues $\tilde{\lambda}_k$, $k \in \mathrm{N}$, of the linear problem

$$\begin{cases} (p(x)y''(x))'' - (q(x)y'(x))' = \lambda\tau(x)\omega \, f_0 y(x), \; 0 < x < l, \\ y \in (b.c.). \end{cases} \tag{23}$$

as represented as follows:

$$\tilde{\lambda}_k = \frac{\lambda_k}{\omega f_0}, \; k \in \mathrm{N}. \tag{24}$$

By (13) we have

$$g_0(y) = f_\infty y - f_0 y + g_\infty(y), \tag{25}$$

and consequently, (22) can be rewritten in the following equivalent form

$$\begin{cases}(p(x)y''(x))''-(q(x)y'(x))'=(\lambda f_0+f_\infty-f_0)\,\tau(x)\,\omega\, y(x)+\tau(x)\omega\; g_\infty(y(x)),\ 0<x<l,\\ y\in(b.c.).\end{cases} \tag{26}$$

Note that the eigenvalues $\hat{\lambda}_k,\ k\in\mathrm{N},$ of the linear problem obtaining from (26) by setting $g_\infty\equiv 0$ has the form

$$\hat{\lambda}_k=\frac{\lambda_k}{\omega f_0}-\frac{f_\infty}{f_0}+1,\ k\in\mathrm{N}. \tag{27}$$

By Lemma 1, Remark 1, [14, Theorem 3.3] and representations (24), (27) it follows from Theorems A and B that the following global bifurcation results hold for the nonlinear eigenvalue problem (22).

**Theorem 1.** *For each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *there exists a continuum* $\tilde{C}_k^\nu$ *of nontrivial solutions of problem (22) which meets* $\left(\frac{\lambda_k}{\omega f_0},0\right)$, *lies in* $R\times S_k^\nu$ *and either* $\tilde{C}_k^\nu$ *meets* $(\lambda,\infty)$ *for some* $\lambda\in R,$ *or the projection of* $\tilde{C}_k^\nu$ *onto* $R\times\{0\}$ *is unbounded.*

**Theorem 2.** *For each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *there exists a continuum* $\tilde{D}_k^\nu$ *of nontrivial solutions of problem (22) which meets* $\left(\frac{\lambda_k}{\omega f_0}-\frac{f_\infty}{f_0}+1,\infty\right)$, *lies in* $R\times S_k^\nu$ *and either* $\tilde{D}_k^\nu$ *meets* $(\lambda,0)$ *for some* $\lambda\in R,$ *or the projection of* $\tilde{D}_k^\nu$ *onto* $R\times\{0\}$ *is unbounded.*

The following result is important in what follows.

**Theorem 3.** *The projections of sets* $\tilde{C}_k^\nu$ *and* $\tilde{D}_k^\nu$ *onto* $R\times\{0\}$ *are bounded.*

**Proof.** We will prove the statement of the theorem for the set $\tilde{C}_k^\nu$, since for the set $\tilde{D}_k^\nu$ the proof can be carried out similarly.

Suppose the opposite, i.e., let for some $k\in\mathrm{N}$ the set $\tilde{C}_k^\nu$ be unbounded. Then there exists $\{(\mu_n,\vartheta_n)\}_{n=1}^\infty\subset\tilde{C}_k^\nu$ such that

$$\lim_{n\to\infty}\mu_n=\infty. \tag{28}$$

Hence we have the following relations:

$$\begin{cases}(p(x)\,\vartheta_n''(x))'' - (q(x)\,\vartheta_n'(x))' = \mu_n \tau(x)\omega f_0\,\vartheta_n(x) + \tau(x)\,\omega\, g_0(\vartheta_n(x)),\; 0 < x < l, \\ \vartheta_n \in (b.c.).\end{cases} \tag{29}$$

Let

$$\varphi_{n,0}(x) = \begin{cases} -\dfrac{g_0(\vartheta_n(x))}{\vartheta_n(x)} & if \quad \vartheta_n(x) \neq 0, \\ 0 & if \quad \vartheta_n(x) = 0. \end{cases} \tag{30}$$

In view of (30) by (29) we get

$$\begin{cases}(p(x)\,\vartheta_n''(x))'' - (q(x)\,\vartheta_n'(x))' + \tau(x)\,\omega\,\varphi_{n0}(x)\,\vartheta_n(x) = \mu_n \tau(x)\,\omega\, f_0\,\vartheta_n(x)\;,\; 0 < x < l, \\ \vartheta_n \in (b.c.).\end{cases} \tag{31}$$

Let $\varepsilon > 0$ be fixed. By (16) it follows from (25) that

$$\frac{|g_0(y)|}{|y|} < <|f_\infty - f_0| + \varepsilon \text{ for any } y \in R, |y| > \Delta_\varepsilon. \tag{32}$$

It is obvious that the function $\dfrac{g_0(y)}{y}$ is continuous on $\{y : \sigma_\varepsilon < |y| < \Delta_\varepsilon\}$, and consequently, there is a positive constant $\kappa_{2,\varepsilon} > 0$ such that

$$\frac{|g_0(y)|}{|y|} < \kappa_{2,\varepsilon} \text{ for any } y \in R,\; \sigma_\varepsilon < |y| < \Delta_\varepsilon. \tag{33}$$

Let

$$k_{3,\varepsilon} = \max\{\,|f_\infty - f_0| + \varepsilon, k_{2,\varepsilon}\}.$$

Then it follows from (15), (32) and (33) that

$$\frac{|g_0(y)|}{|y|} < \kappa_{3,\varepsilon} \text{ for any } y \in R,\; y \neq 0. \tag{34}$$

Hence in view of (34), by Lemma 4.1 and Remark 4.1, from (31) we obtain

$$|\mu_n - \lambda_k| \leq \kappa_{3\varepsilon} \text{ for any } n \in \mathrm{N},$$

which contradict relation (28). The proof of this theorem is complete.

It follows from Theorems 1-3 and [3, Theorem 3.3] the following result.

**Corollary 1.** *For each* $k \in \mathrm{N}$ *and each* $\nu \in \{+,-\}$ *the relation*

$$\tilde{C}_k^\nu = \tilde{D}_k^\nu. \tag{35}$$

The following theorem is the main result of this paper.

**Theorem 4.** *Let for some* $k \in \mathrm{N}$ *the following condition holds:*

$$\frac{\lambda_k}{h_0} < \omega < \frac{\lambda_k}{h_\infty} \text{ or } \frac{\lambda_k}{h_\infty} < \omega < \frac{\lambda_k}{h_0}. \quad (36)$$

*Then for each* $\nu \in \{+,-\}$ *there exist a solution* $\vartheta_k^\nu$ *of problem (1)-(3) such that* $\vartheta_k^\nu$ *has exactly* $k-1$ *simple nodal zeros in the interval* $(0, l)$, *or more precisely* $\vartheta_k^\nu \in S_k^\nu$.

**Proof.** If $\lambda_k = 0$, then the result is trivial. Indeed, in this case $k = 1$ and $\omega = 0$, consequently, problem (1)-(5) has two solution $\vartheta_1^+$ and $\vartheta_1^-$ which have no zeros in the interval $(0, l)$ (see [3]).

Now let

$$\lambda_k > 0 \text{ and } \frac{\lambda_k}{f_0} < \omega < \frac{\lambda_k}{f_\infty} \text{ for some } k \in \mathrm{N}. \quad (37)$$

Then it follows from (37) that

$$\frac{\lambda_k}{\omega f_0} < 1. \quad (38)$$

Moreover, in view of (38) by (37) we get

$$\frac{\lambda_k}{\omega f_0} - \frac{f_\infty}{f_0} > \frac{\lambda_k}{\frac{\lambda_k}{f_\infty} f_0} - \frac{f_\infty}{f_0} = \frac{f_\infty}{f_0} - \frac{f_\infty}{f_0} = 0. \quad (39)$$

By (38) and (39) we have the following relation

$$\frac{\lambda_k}{\omega f_0} < 1 < \frac{\lambda_k}{\omega f_0} - \frac{f_\infty}{f_0}. \quad (40)$$

It follows from Theorems 1-3 and Corollary 1 that $\tilde{C}_k^\nu = \tilde{D}_k^\nu \subset R \times S_k^\nu$, the set $\tilde{C}_k^\nu$ is connected and by [14, Theorem 3.3] this set meets both points $\left(\frac{\lambda_k}{f_0}, 0\right)$ and $\left(\frac{\lambda_k}{f_0} - \frac{f_\infty}{f_0} + 1, \infty\right)$. Therefore, by (40) the set $\tilde{C}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in the space $R \times E$, and consequently, for each $\nu \in \{+,-\}$ there exists $\vartheta_k^\nu \in S_k^\nu$ which is a solution to problem (22) for $\lambda = 1$, i.e., this function is a solution of original problem (1)-(5).

Similarly, it can be shown that the statement of this theorem holds in the case where the second condition of (36) is satisfied. The proof of the theorem is complete.

## References

[1] Aliyev ZS. Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, Sb. Math. **2016**, v. 207 (12), p. 1625-1649.

[2] Z.S. Aliyev ZS, Mustafayeva NA. Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, Electron. J. Differ. Equ. **2018**, (98), p. 1-19.

[3] Banks DO, Kurowski G.J. A Prufer transformation for the equation of a vibrating beam subject to axial forces, J. Differential Equations **1977**, v. 24 (1), p. 57--74.

[4] Bolotin BB (ed.), Vibrations in technology. Handbook in 6 volumes. Vol. 1. Vibrations of linear systems, Mashinostroenie, Moscow, 1978 [in Russian].

[5] Cantrell RS, Cosner C. Spatial ecology via reaction- diffusion equations, Wiley, Chichester, 2003.

[6] Dalbono F, Zanolin F. Multiplicity results for asymptotically linear equations, Using the rotation number approach. Mediterr. J. Math. **2007,** v. 4, p. 127-149.

[7] Ma R, Dai G. Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, J. Funct. Anal. **2013,** v. 265 (8), p. 1443-1459.

[8] Ma R, Thompson B. Nodal solutions for nonlinear eigenvalue problems, Nonlinear Anal. **2004,** v. 59 (5), p. 707-718.

[9] Ma R, Thompson B. Multiplicity results for second-order two-point boundary value problems with nonlinearities across several eigenvalues, Appl. Math. Lett. 2005, v. 18 (5), p. 587-595.

[10] Ma R, Gao C. Nodal solutions of a nonlinear eigenvalue problem of the Euler-Bernoulli equation. J. Math. Anal. Appl. **2012**, v. 387 (2), p. 1160-1166.

[11] Ma R, Thompson B. Nodal solutions for a nonlinear fourth-order eigenvalue problem, Acta Math. Sin. Engl. Ser. 2008, v. 24 ( 1), p. 27-34

[12] Naito Y, Tanaka S. On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, Nonlinear Anal. **2004,** 56 (4), p. 919-935.

[13] Roseau M. Vibrations in Mechanical Systems, Analytical Methods and Applications. Springer, Berlin, **1987**.

[14] Rynne BP. Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, J. Math. Anal. Appl. 1998, v. 228 (1), p. 141-156.

[15] Tanaka S. On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem. Differential Integral Equations **2007,** v. 20 (1), p. 93-104.