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BASIS PROPERTIES OF ROOT FUNCTIONS OF THE EIGENVALUE PROBLEM FOR THE EQUATION OF A VIBRATING BEAM WITH A SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

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Abstract

In this paper, we consider the eigenvalue problem for the equation of a vibrating beam with a spectral parameter contained in the boundary conditions. The general characteristics of the location of eigenvalues on the real axis are studied, the multiplicities of eigenvalues are found, and the oscillatory properties of the eigenfunctions of this spectral problem are investigated. Moreover, asymptotic formulas for the eigenvalues are obtained and sufficient conditions are established for the subsystems of root functions to form a basis in Lebesgue spaces.

Keywords: eigenvalue problem, spectral parameter, multiple of the eigenvalue, oscillatory properties, basis properties

Mathematics Subject Classification *(2020): 34A30, 34B05, 34B09, 34C23; 34L10, 47A75, 74H45*

1. Introduction

We consider the following spectral problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1}$$

$$y''(0) = 0, \tag{2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{3}$$

$$y''(1) - b\lambda y'(1) = 0, \tag{4}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{5}$$

where $\lambda \in R$ is a spectral parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on the interval $[0, 1]$, a, b, c are positive constants.

This problem describes small bending vibrations of a homogeneous beam of constant stiffness, in the cross sections of which a longitudinal force acts, at the left end of which a tracking force acts, and at the right end the inertial load is concentrated (see, for example, [9, 19, 20]).

Eigenvalue problems for ordinary differential equations with a spectral parameter in boundary conditions have been considered by many authors in various formulations (see [1-6, 10, 11, 13-18, 21]). The basis properties in various function spaces of root functions of the Sturm-Liouville problems with a spectral parameter in boundary conditions were studied in the papers [2, 10, 13-15, 21] (see also their bibliography). The oscillation properties of eigenfunctions and basis properties in $L_p, 1 < p < \infty$, of root functions of eigenvalue problems for ordinary differential equations of fourth order with a spectral parameter in boundary conditions were studied in [1, 3-6, 11, 16-18]. In papers [1, 3-6, 17, 18], the authors establish necessary and sufficient conditions, as well as sufficient conditions for the system of root functions of considered problems to form a defect basis (with a finite number of defects) in the space $L_p, 1 < p < \infty$.

This paper is a continuation of the research that began in the above-mentioned papers. Here studied the oscillatory properties of the eigenfunctions and their derivatives, obtained asymptotic formulas for eigenvalues and eigenfunctions, establish necessary and sufficient conditions, as well as sufficient conditions for the subsystems of root functions of problem (1)-(5) to form a basis in the space $L_p(0, 1), 1 < p < \infty$.

2. Properties of eigenvalues of problem (1)-(5)

Lemma 1. *The eigenvalues of problem (1)-(5) are real.*

Proof. By following the arguments in Theorem 3.1 of [17] we can show that for each $\lambda \in C$ there exists a unique nontrivial solution $y(x,\lambda)$ of problem (1), (2), (4), (5) up to a constant factor. Then each nonzero eigenvalue of problem (1)-(5) is a root of the following equation

$$Ty(0,\lambda) - a\lambda y(0,\lambda) = 0. \tag{7}$$

Let $\lambda \in C \setminus R$ be the eigenvalue of problem (1)-(5). Then $\bar{\lambda}$ is also an eigenvalue of this problem because the coefficients of the equation and the boundary conditions are real. In this case $y(x,\bar{\lambda}) = \overline{y(x,\lambda)}$ is eigenfunction of problem (1)-(5) corresponding to the eigenvalue $\bar{\lambda}$.

It follows from [1, formula (3.4)] that

$$\int_0^1 |\, y(x,\lambda)|^2\, dx + a\,|\, y(0,\lambda)|^2 + b\,|\, y'(1,\lambda)|^2 - c\,|\, y(1,\lambda)|^2 = 0. \tag{8}$$

Next, multiplying both sides of (1.1) by $\overline{y(x,\lambda)}$, integrating the resulting equality in the range from 0 to 1, using the formula for integration by parts and taking into account (2)-(5), we get

$$\int_0^1 \left\{ |\, y''(x,\lambda)|^2 + q(x)\,|\, y'(x,\lambda)|^2 \right\} dx =$$

$$\lambda \left\{ \int_0^1 |\, y(x,\lambda)|^2\, dx + a\,|\, y(0,\lambda)|^2 + b\,|\, y'(1,\lambda)|^2 - c\,|\, y(1,\lambda)|^2 \right\}. \tag{9}$$

By (8) it follows from (9) that

$$\int_0^1 \left\{ |\, y''(x,\lambda)|^2 + q(x)\,|\, y'(x,\lambda)|^2 \right\} dx = 0.$$

From the last relation we obtain $y'(x,\lambda) \equiv 0$, and consequently,

$$y(x,\lambda) \equiv const \neq 0,$$

which contradicts Eq. (1). The proof of this lemma is complete.

Lemma 2. *The nonzero eigenvalues of problem (1)-(5) are simple.*

Proof. Let $\lambda \in R,\ \lambda \neq 0,$ be the double eigenvalue of problem (1)-(5). Then by (7) we have

$$y''(1,\lambda) - b\lambda y'(1,\lambda) = 0 \ \text{ and } \ \frac{\partial y''(1,\lambda)}{\partial \lambda} - b\lambda \frac{\partial y'(1,\lambda)}{\partial \lambda} - by'(1,\lambda) = 0. \tag{10}$$

By [1, formula (2.21)] for any $\lambda, \mu \in R \setminus \{0\}$ we have

$$-\frac{y''(1,\mu) - y''(1,\lambda)}{\mu - \lambda}\, y'(1,\lambda) + y''(1,\lambda)\, \frac{y'(1,\mu) - y'(1,\lambda)}{\mu - \lambda} =$$

$$\int_0^1 y(x,\mu)y(x,\lambda)\,dx + a\,y(0,\mu)y(0,\lambda) + by'(1,\mu)y'(1,\lambda) - cy(1,\mu)y(1,\lambda). \tag{11}$$

Passing to the limit in (11) as $\mu \to \lambda$, we obtain

$$-\frac{\partial y''(1,\lambda)}{\partial\lambda}y'(1,\lambda) + y''(1,\lambda)\frac{\partial y'(1,\lambda)}{\partial\lambda} =$$

$$\int_0^1 y^2(x,\lambda)\,dx + a\,y^2(0,\lambda) - cy^2(1,\lambda). \tag{12}$$

By (11) we get

$$y''(1,\lambda) = b\lambda y'(1,\lambda) \quad \text{and} \quad \frac{\partial y''(1,\lambda)}{\partial\lambda} = b\lambda\frac{\partial y'(1,\lambda)}{\partial\lambda} + by'(1,\lambda). \tag{13}$$

Using (13) from (12) we find the following relation

$$\int_0^1 y^2(x,\lambda)\,dx + a\,y^2(0,\lambda) + by'^2(1,\lambda) - cy^2(1,\lambda). \tag{14}$$

Since $\lambda \in R$, $\lambda \neq 0$, according (14) from (9) we obtain

$$\int_0^1 \{y''^2(x,\lambda) + q(x)y'^2(x,\lambda)\}\,dx = 0,$$

whence implies that $y(x,\lambda) \equiv const \neq 0$, which contradicts Eq. (1). The proof of this lemma is complete.

3. Operator treatment of problem (1)-(5)

Let H be a Hilbert space $L_2(0,1) \oplus C^3$ with the following scalar product

$$(\hat{y},\hat{\vartheta})_H = \int_0^1 y(x)\overline{\vartheta(x)}\,dx + a^{-1}m\bar{s} + b^{-1}n\bar{t} + c^{-1}l\,\bar{r}, \tag{15}$$

where

$$\hat{y} = \{y,m,n,l\},\ \hat{\vartheta} = \{\vartheta,s,t,r\}.$$

We define the operator

$$L\hat{y} = L\{y,m,n,l\} = \{\ell(y), Ty(0), y''(1), Ty(1)\}$$

on the domain

$$D(L) = \{\hat{y} = \{y,m,n,l\} \in H:\ y \in W_2^4(0,1), \ell(y) \in L_2(0,1), m = a\,y(0),$$
$$n = by'(1), l = cy(1)\},$$

which is dense everywhere in H. Then problem (1)-(5) is reduced to the following operator equation

$$L\hat{y} = \lambda\hat{y},\ \hat{y} \in D(L). \tag{16}$$

In this case, the eigenvalues $\lambda_k, k \in \mathrm{N}$, of problems (1)-(5) and (6) coincide with each other (counting multiplicities), and there exists a one-to-one correspondence between the root functions of problem (1)-(5) and the root vectors of problem (6),

$$y_k \leftrightarrow \hat{y}_k = \{y_k, m_k, n_k, l_k\},\ m_k = a\,y_k(0),\ n_k = by'_k(1), l_k = cy_k(1).$$

In the case of $a > 0, b > 0$ and $c < 0$, by [5, Theorem 4.1], the operator L is a self-adjoint in the space H and the system of eigenvectors of this operator forms an orthogonal basis in H.

In the case of $a > 0, b > 0$ and $c > 0$ it is easy to see that the operator L is not symmetric in H. In this case we introduce the operator $J : H \to H$ defined by

$$J\{y, m, n, l\} = \{y, m, n, -l\}.$$

It is obvious that the operator J is unitary and symmetric in H, and its spectrum consists of two eigenvalues: -1 of multiplicity one and $+1$ of infinite multiplicity. Then this operator generates the Pontryagin space $\Pi_1 = L_2(0,1) \oplus C^3$ with the inner product

$$(\hat{y}, \hat{\vartheta})_{\Pi_1} = (J\hat{y}, \hat{\vartheta})_H = (\{y, m, n, -l\}, \{\vartheta, s, t, r\})_H = \int_0^1 y(x)\,\overline{\vartheta(x)}\,dx + \\ + a^{-1}m\bar{s} + b^{-1}n\,\bar{t} - c^{-1}l\,\bar{r}, \tag{17}$$

It follows from [7, Section 3, Propositions 1^0 and 5^0; Section 4, Theorem 4.2] that

1^0. the operator L is J -self-adjoint in Π_1;

2^0. the operator L^* conjugate to operator L in H has the representation $L^* = JLJ$;

3^0. the system of root vectors $\{y_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k, l_k\}$, $m_k = a\,y_k(0)$, $n_k = by'_k(1)$, $l_k = cy_k(1)$, corresponding to the system $\{\lambda_k\}_{k=1}^{\infty}$, of eigenvalues of the operator L, forms an unconditional basis in H.

4. Properties of solutions of initial-boundary value problem (1), (2), (4), (5)

By following the arguments in [4] we can prove the following result.

Theorem 1. *The eigenvalues of the spectral problem (1), (4), (5) and*

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \qquad (18)$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \qquad (19)$$

where $\alpha, \beta \in [0, \pi/2]$, *are real, with the exception of the case* $\beta = \pi/2$ *and* $c = 1$, *when the eigenvalue* $\lambda = 0$ *has algebraic multiplicity* 2, *and form an infinitely non-decreasing sequence* $\{\lambda_k(\alpha,\beta)\}_{k=1}^{\infty}$ *such that*

$$\lambda_1(\alpha,\beta) < 0 < \lambda_2(\alpha,\beta) < \ldots < \lambda_k(\alpha,\beta) < \ldots \quad \text{for} \quad \beta \in [0, \pi/2),$$

$$\lambda_1(\alpha,\pi/2) < 0 = \lambda_2(\alpha,\pi/2) < \ldots < \lambda_k(\alpha,\pi/2) < \ldots \quad \text{for} \quad c < 1,$$

$$\lambda_1(\alpha,\pi/2) = 0 = \lambda_2(\alpha,\pi/2) < \ldots < \lambda_k(\alpha,\pi/2) < \ldots \quad \text{for} \quad c = 1,$$

$$0 = \lambda_1(\alpha,\pi/2) < \lambda_2(\alpha,\pi/2) < \ldots < \lambda_k(\alpha,\pi/2) < \ldots \quad \text{for} \quad c > 1.$$

Let $\lambda_k(0) = \lambda_k(\pi/2, 0)$ and $\lambda_k(\pi/2) = \lambda_k(\pi/2, \pi/2)$. Then by [4, Theorem 2.3] we get

$$\begin{gathered}\lambda_1(0) < \lambda_1(\pi/2) < 0 = \lambda_2(\pi/2) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(0) < \ldots \quad \text{for} \quad c < 1, \\ \lambda_1(0) < \lambda_1(\pi/2) = 0 = \lambda_2(\pi/2) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(0) < \ldots \quad \text{for} \quad c = 1, \\ \lambda_1(0) < \lambda_1(\pi/2) = 0 < \lambda_2(\pi/2) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(0) < \ldots \quad \text{for} \quad c > 1.\end{gathered} \qquad (20)$$

Let $y(x,\lambda)$ be the solution of problem (1), (2), (4), (5). It follows from Remark 3.1 of [17] that this function is an entire function of the variable λ for each fixed $x \in [0, 1]$. Then it is obvious that the function

$$F(\lambda) = \frac{Ty(0,\lambda)}{y(0,\lambda)}$$

is defined in the set

$$B = (C \setminus R) \cup \bigcup_{i=1}^{\infty} (\lambda_{k-1}(0), \lambda_k(0)),$$

where $\lambda_0(0) = -\infty$, and is meromorphic function. In this case $\lambda_k(0)$ and $\lambda_k(\pi/2)$ are poles and zeros of this function, respectively.

Following the corresponding reasoning carried out in the proof of [3, Lemmas 3.3 and 3.4] and [4, Lemma 3.3], we can show that

$$\frac{dF}{d\lambda} = -\frac{1}{y^2(0,\lambda)}\left\{\int_0^1 y^2(x,\lambda)dx + by'^2(1,\lambda) - cy^2(1,\lambda)\right\}, \tag{21}$$

$$\lim_{\lambda\to-\infty} F(\lambda) = +\infty, \tag{22}$$

$$\begin{gathered}\lim_{\lambda\to\lambda_1(0)-0} F(\lambda) = +\infty,\ \lim_{\lambda\to\lambda_1(0)+0} F(\lambda) = -\infty,\\ \lim_{\lambda\to\lambda_k(0)-0} F(\lambda) = -\infty,\ \lim_{\lambda\to\lambda_k(0)+0} F(\lambda) = +\infty,\ k \ge 2,\end{gathered} \tag{23}$$

Moreover, for the function $F(\lambda)$ the following representation holds:

$$F(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda c_k}{\lambda_k(0)(\lambda - \lambda_k(0))}, \tag{24}$$

where $c_k = \underset{\lambda=\lambda_k(0)}{res} F(\lambda),$ and $c_1 < 0$ and $c_k > 0$ for $k = 2, 3, \ldots$.

Remark 1. It follows from (24) that

$$F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3},$$

whence implies that $F''(\lambda) < 0$ for $\lambda \in (\lambda_1(0), \lambda_2(0)),$ i.e., the function $F(\lambda)$ is concave in the interval $(\lambda_1(0), \lambda_2(0)).$

Now we study oscillatory properties of the function $y(x,\lambda)$ for $\lambda \in R.$

Lemma 4. *The zeros of the function* $y(x,\lambda),$ $\lambda \in R,$ *contained in* $[0, 1)$ *and of the function* $y'(x,\lambda),$ $\lambda \in R,$ $\lambda \ne 0,$ *contained in* $(0, 1),$ *are simple and continuously differentiable functions of the parameter* $\lambda.$

The proof of this lemma is similar to that of [17, Lemmas 3.2 and 3.9].

Corollary 1. *As* λ *varies, the functions* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *can lose zeros or gain zeros only by these zeros leaving or entering the interval* $[0, 1]$ *through its right endpoint* $x = 0$ *for* $\lambda > 0,$ *and left endpoint* $x = 1$ *for* $\lambda < 0,$ *respectively.*

Let $\chi(\lambda)$ and $\tau(\lambda)$ be the number of zeros contained in $(0, 1)$ of functions $y(x,\lambda)$ and $y'(x,\lambda),$ respectively.

Lemma 5. *Let* $\lambda > 0.$ *Then we have the following assertions: (i)* $\chi(\lambda) = 0$ if $\lambda \in (0, \lambda_2(0)],$ *and* $\chi(\lambda) = k-2$ if $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *for* $k = 3, 4, \ldots$; *(ii)* $\tau(\lambda) = 0$ if $\lambda \in (0, \lambda_2(0)],$ *and* $\tau(\lambda) = k-3$ if $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2)],$ $\tau(\lambda) = k-3$ or $\tau(\lambda) = k-2$ if $\lambda \in (\lambda_k(\pi/2), \lambda_k(0)]$ *for* $k = 3, 4, \ldots$.

The proof of this lemma is similar to that of [17, Theorem 3.2] by the use of Lemma 4 and Corollary 1.

Remark 2. In the case of $\lambda < 0$, the number of zeros of functions $y(x, \lambda)$ and $y'(x, \lambda)$ contained in the interval $(0, 1)$ can be determined as in [3, Section 3, pp. 9-10]. Should be noted that the number of zeros of these functions can be arbitrary, depending on the location of the parameter λ on the negative axis.

5. Properties of eigenvalues and eigenfunctions of problem (1)-(5)

Remark 3. If $\lambda = 0$, then it follows from (1)-(5) that

$$(Ty)'(x) = 0, \ \ 0 < x < 1, \tag{25}$$

$$y''(0) = 0, \ Ty(0) = 0, \ \ y''(1) = 0, \ Ty(1) = 0. \tag{26}$$

Hence we have $Ty \equiv 0$. Let $\varphi_k(x), \ \ k = \overline{1, 4}$, be the solutions of Eq. (1) that satisfy the Cauchy conditions (normalized for $x = 1$)

$$\varphi_k^{(s-1)}(1) = \delta_k^s, \ s = 1, 2, 3, \ \ T\varphi_k(1) = \delta_k^4, \tag{27}$$

where δ_k^s is the Kronecker delta. Obviously, the solution $y_0(x)$ of problem (25), (26) can be expressed as

$$y_0(x) = C_1\varphi_1(x) + C_2\varphi_2(x) + C_3\varphi_3(x) + C_4\varphi_4(x), \tag{28}$$

where $C_k, \ \ k = \overline{1, 4}$, are some constants.

In view of conditions $y_0''(1) = 0$ and $Ty_0(1) = 0$, by (27) it follows from (28) that $C_3 = C_4 = 0$.

It is easy see that $\varphi_1 \equiv 1$, and consequently,

$$y_0(x) = C_1 + C_2\varphi_2(x). \tag{29}$$

From (29) we get

$$y_0''(1) = C_2\varphi_2''(1) = 0.$$

By following the arguments in Remark 2.2 of [1] we can show that $\varphi_2''(1) \neq 0$, and consequently, $C_2 = 0$. Therefore, $y \equiv const$ is a solution of problem (25), (26), or more precisely, $y_0 \equiv const$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$ of problem (1)-(5) (without loss of generality we can assume that $y_0 \equiv 1$).

Remark 4. Now we consider the following boundary value problem

$$\begin{cases} (T\vartheta)'(x) = \lambda\vartheta(x) + y(x),\ 0 < x < 1, \\ \vartheta''(0) = 0, \\ T\vartheta(0) - a\lambda\vartheta(0) - ay(0) = 0, \\ \vartheta''(1) - b\lambda\vartheta'(1) - b\lambda y'(1) = 0, \\ T\vartheta(1) - c\lambda\vartheta(1) - cy(1) = 0, \end{cases} \tag{30}$$

For $\lambda = 0$ from (30) we get

$$(T\vartheta)'(x) = 1,\ 0 < x < 1, \tag{31}$$

$$\vartheta''(0) = 0,\ T\vartheta(0) - a = 0, \tag{32}$$

$$\vartheta''(1) = 0,\ T\vartheta(1) - c = 0. \tag{33}$$

It follows from (31) that

$$T\vartheta(x) = x + \tau,$$

where τ is some real constant. Then by the second relation of (32) we have $\tau = a,$ and consequently, by the second condition of (33) we have $1 + a - c = 0.$ Thus we have $\lambda = 0$ is a simple eigenvalue of problem (1)-(5) for $c \neq a+1,$ is a double eigenvalue for $c = a+1$ and it corresponds to the chain consisting of the eigenfunction $y_0(x)$ and the associated function $\vartheta_0(x).$

Remark 5. By Lemmas 2 and 3 the eigenvalues (taking into account their multiplicities) of problem (1)-(5) are roots of the equation

$$Ty(0,\lambda) - a\lambda y(0,\lambda) = 0. \tag{34}$$

If $y(0,\lambda) = 0$ for some $\lambda \in R,\ \lambda \neq \lambda_k(0),$ then it follows from (34) that $Ty(0,\lambda) = 0.$ Hence λ is an eigenvalue of problem (1), (2), (19), (4), (5) as for $\beta = 0,$ also for $\beta = \pi/2$ in contradiction with relations in (20). Therefore, the nonzero roots of Eq. (4.1) are also the roots of the following equation

$$F(\lambda) - a\lambda = 0. \tag{35}$$

Lemma 6. *In the interval* $(-\infty, \lambda_1(0)),$ *Eq. (35) has no roots.*

The proof of this lemma follows directly from relations (20), (22) and (23) taking into account the condition $a > 0.$

Lemma 7. *For each* $k \in \mathrm{N},\ k \geq 3,$ *in* $(\lambda_{k-1}(0), \lambda_k(0)),$ *Eq. (35) cannot have more than one root.*

Proof. Let Eq. (35) has two distinct roots $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ in the interval $(\lambda_{k-1}(0), \lambda_k(0))$ for some natural $k_0 \geq 3$ (without loss of generality we can assume that $\tilde{\lambda}_1 < \tilde{\lambda}_2$). Then there exists $\tilde{\lambda}_0 \in (\tilde{\lambda}_1, \tilde{\lambda}_2)$ such that

$$F'(\tilde{\lambda}_0) - a = 0.$$

Hence by (21) we get

$$-\frac{1}{y^2(0, \tilde{\lambda}_0)} \left\{ \int_0^1 y^2(x, \tilde{\lambda}_0) dx + by'^2(1, \tilde{\lambda}_0) - cy^2(1, \tilde{\lambda}_0) \right\} - a = 0,$$

which implies that

$$\int_0^1 y^2(x, \tilde{\lambda}_0)\, dx + ay^2(0, \tilde{\lambda}_0) + by'^2(1, \tilde{\lambda}_0) - cy^2(1, \tilde{\lambda}_0) = 0. \tag{36}$$

Putting $y(x, \tilde{\lambda}_0)$ instead of $y(x)$ in (1), multiplying both sides of the resulting equality by $y(x, \tilde{\lambda}_0)$, integrating this equality in the range from 0 to 1, applying the formula for integration by parts and taking into account the boundary conditions (2)-(5) we obtain

$$\begin{gathered} \int_0^1 \{y''^2(x, \tilde{\lambda}) + q(x) y'^2(x, \tilde{\lambda})\} dx = \\ \int_0^1 y^2(x, \tilde{\lambda}_0)\, dx + ay^2(0, \tilde{\lambda}_0) + by'^2(1, \tilde{\lambda}_0) - cy^2(1, \tilde{\lambda}_0) = 0. \end{gathered} \tag{37}$$

Taking (36) into account, from relation (37) we obtain

$$\int_0^1 \{y''^2(x, \tilde{\lambda}) + q(x) y'^2(x, \tilde{\lambda})\} dx = 0. \tag{38}$$

Then from (38) it follows that $y'(x, \tilde{\lambda}) \equiv 0$, which contradicts (1). The proof of this lemma is complete.

Theorem 2. *The eigenvalues of problem (1)-(5) form an infinitely non-degreasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that*

$$\lambda_1 \leq \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots,$$

and have the following location on real axis:

$$\lambda_1, \lambda_2 \in (\lambda_1(0), \lambda_2(0)), \lambda_3 \in (\lambda_2(0), \lambda_3(0)), \ldots, \lambda_k \in (\lambda_{k-1}(0), \lambda_k(0)), \ldots,$$

$$\lambda_1 < 0 = \lambda_2 \text{ for } c \leq 1 \text{ and } c > 1,\ a > c - 1,$$

$$\lambda_1 = 0 < \lambda_2 \text{ for } c > 1,\ a < c - 1,$$

$$\lambda_1 = 0 = \lambda_2 \text{ for } c > 1,\ a = c - 1.$$

Moreover, for every $k \in \mathrm{N}$ *the eigenfunction* $y_k(x)$ *corresponding to the eigenvalue* λ_k *and the function* $y_k'(x)$ *have the following oscillation properties:*

(i) the function $y_k(x)$ *for* $k \geq 3$ *has exactly* $k-2$ *simple zeros in* $(0, 1)$, *the function* $y_2(x)$ *has no zeros in* $(0, 1)$ *for* $c \leq 1$ *and* $c > 1,\ a \neq c-1,$ *the function* $y_1(x)$ *has no zeros in* $(0, 1)$ *for* $c > 1,\ a \leq c-1$ *and can have an arbitrary number of zeros in* $(0, 1)$ *for* $c \leq 1$ *and* $c > 1,\ a > c-1;$

(ii) the function $y_k'(x)$ *for* $k \geq 3$ *has exactly* $k-3$ *simple zeros in* $(0, 1)$, $y_2'(x) \equiv 0$ *for* $c \leq 1$ *and* $c > 1,\ a > c-1,$ $y_2'(x)$ *has no zeros in* $(0, 1)$ *for* $c > 1,\ a < c-1,$ $y_1'(x) \equiv 0$ *for* $c > 1,\ a \leq c-1,$ *and* $y_1'(x)$ *can have an arbitrary number of zeros in* $(0, 1)$ *for* $c \leq 1$ *and* $c > 1,\ a > c-1.$

Proof. By (35) (see Remark 5) the eigenvalues (taking into account their multiplicities) of problem (1)-(5) are roots of the equation

$$F(\lambda) = a\lambda. \tag{39}$$

By Lemma 3 in the interval $(-\infty, \lambda_1(0))$ Eq. (39) has no roots, and consequently, problem (1)-(5) has no eigenvalues in the same interval.

By (20), (21), (23) and Remark 3 we have the following relations

$$\lambda_1(0) < \lambda_1(\pi/2) < 0 = \lambda_2(\pi/2) < \lambda_2(0) \quad \text{for} \quad c < 1, \tag{40}$$

$$\lambda_1(0) < \lambda_1(\pi/2) = 0 = \lambda_2(\pi/2) < \lambda_2(0), \quad \text{for} \quad c = 1, \tag{41}$$

$$\lambda_1(0) < \lambda_1(\pi/2) = 0 < \lambda_2(\pi/2) < \lambda_2(0) \quad \text{for} \quad c > 1. \tag{42}$$

$$\lim_{\lambda \to \lambda_1(0)+0} F(\lambda) = -\infty,\ F(\lambda_1(\pi/2)) = F(\lambda_2(\pi/2)) = 0,\ \lim_{\lambda \to \lambda_2(0)-0} F(\lambda) = -\infty. \tag{43}$$

$$F'(0) = c-1. \tag{44}$$

Moreover, by Remark 1, the function $F(\lambda)$ is concave on the interval $(\lambda_1(0), \lambda_2(0))$.

If $c \leq 1$, then in view of (44) we get

$$F'(0) < 0 \quad \text{for} \quad c < 1 \quad \text{and} \quad F'(0) = 0 \quad \text{for} \quad c = 1. \tag{45}$$

Since $a > 0$ and $F(\lambda)$ is concave on $(\lambda_1(0), \lambda_2(0))$ by (45) the graph of the function $G(\lambda) = a\lambda$ no tangent to the graph of the function $F(\lambda)$ at the point $\lambda = 0$. Then, according (40) and (43), the graph of the function $G(\lambda)$ intersects the graph of the function $F(\lambda)$ for $c \leq 1$ at two points $\lambda_1(0) < \lambda_1 < 0$ and $\lambda_2 = 0$.

If $c>1$ and $a\neq c-1,$ then the graph of the function $G(\lambda)$ no tangent to the graph of the function $F(\lambda)$ at the point $\lambda=0.$ Then in this case the graph of the function $G(\lambda)$ intersects the graph of the function $F(\lambda)$ at two points $\lambda_1(0)<\lambda_1<0$ and $\lambda_2=0$ for $a>c-1,$ and $\lambda_1=0$ and $0<\lambda_2<\lambda_2(0)$ for $a<c-1.$

If $c>1$ and $a=c-1,$ then the graph of the function $G(\lambda)$ tangent to the graph of the function $F(\lambda)$ at the point $\lambda=0.$ In this case $\lambda=0$ is a root of Eq. (39) (also Eq. (35)) with multiplicity two (we believe that $\lambda_1=\lambda_2=0$). By Remark 4, the eigenvalue $\lambda=0$ corresponds to a chain consisting of the eigenfunction $y_0(x)$ and the associated function $\vartheta_0(x).$

It follows from (23) that

$$\lim_{\lambda\to\lambda_{k-0}(0)+0}F(\lambda)=+\infty,\ \lim_{\lambda\to\lambda_k(0)-0}F(\lambda)=-\infty,\ F(\lambda)>0 \text{ if } \lambda\in(\lambda_{k-1}(0),\lambda_k(\pi/2)),$$
$$F(\lambda_k(\pi/2))=0, \text{ and } F(\lambda)<0 \text{ if } \lambda\in(\lambda_k(\pi/2),\lambda_k(0)) \text{ for } k\geq 3. \quad (46)$$

Since $F(\lambda)\in C((\lambda_{k-1}(0),\lambda_k(0)))$ in view of Lemma 7 it follows from (46) that Eq. (45) has unique root λ_k in the interval $(\lambda_{k-1}(0),\lambda_k(0))$ for $k\geq 3.$ It should be noted that in this case $\lambda_k\in(\lambda_{k-1}(0),\lambda_k(\pi/2)).$

The oscillatory properties (i) and (ii) of eigenfunctions of problem (1)-(5) and their derivatives follows from Lemma 5 and Remark 2. The proof of this theorem is complete.

Remark 6. By Theorem 2 we have

$$y_k(x)=y(x,\lambda_k) \text{ for } k=1,3,\ldots, \quad (47)$$

$$y_2(x)=y(x,\lambda_2) \text{ if } c\leq 1 \text{ and } c>1,\ a\neq c-1, \quad (48)$$

$$y_2(x)=y_2^*(x)+d\,y_1(x) \text{ if } c>1,\ a=c-1, \quad (49)$$

where $y_2^*(x)=\dfrac{\partial y(x,\lambda_2)}{\partial\lambda}$ and d is an arbitrary constant.

6. Basis properties of root functions of problem (1)-(5)

It follows from [12] that for each $k\in\mathrm{N}$ the following relation holds:

$$L\hat{y}_k=\lambda_k\hat{y}_k+\theta_k\hat{y}_{k-1}, \quad (50)$$

where $\hat{y}_k=\{y_k,m_k,n_k,l_k\},$ $m_k=a\,y_k(0),$ $n_k=by_k'(1), l_k=cy_k(1),$ $\theta_k=0$ if $\hat{y}_k$ is an eigenvector, $\theta_k = 1$ if $\hat{y}_k$ is an associated vector.

Let $\{\hat{\vartheta}_k^*\}_{k=1}^{\infty}$, $\hat{\vartheta}_k^* = \{\vartheta_k^*, s_k^*, t_k^*, r_k^*\}$, be the system of root vectors of the operator L^*. Then by the formula (3) of [12] we get

$$L^* \hat{\vartheta}_k^* = \lambda_k \hat{\vartheta}_k^* + \theta_{k+1} \vartheta_{k+1}^*. \tag{51}$$

Since the operator L^* has representation $L^* = JLJ$ (see property 2^0 of the operator L) by Remark 6 it follows from (47)- (51) that

$$\hat{\vartheta}_k^* = J\hat{y}_k \text{ for } k = 2, 3, \ldots\ , \tag{52}$$

$$\hat{\vartheta}_1^* = J\hat{y}_1 \text{ if } c \le 1 \text{ and } c > 1,\ a \ne c - 1, \tag{53}$$

$$\hat{\vartheta}_1^* = J\hat{y}_2^* + \hat{d} J y_1 \text{ if } c > 1,\ a = c - 1, \tag{54}$$

where $\hat{y}_2^* = \{y_2^*, m_2^*, n_2^*, l_2^*\}$, $m_2^* = m'(\lambda_2)$, $n_2^* = n'(\lambda_2)$, $l_2^* = l'(\lambda_2)$, and $\hat{d}$ is an arbitrary constant.

Let $k, \chi \in \mathrm{N}$ such that λ_k and λ_χ are simple. Then by (50) and (51) we have

$$L\hat{y}_k = \lambda_k \hat{y}_k \text{ and } L^* \hat{\vartheta}_\chi^* = \lambda_\chi \hat{\vartheta}_\chi^*. \tag{55}$$

Then from (55) we get

$$(L\hat{y}_k, \hat{\vartheta}_\chi^*)_H = \lambda_k (\hat{y}_k, \hat{\vartheta}_\chi^*)_H \text{ and } (y_k, L^* \hat{\vartheta}_\chi^*)_H = \lambda_\chi (\hat{y}_k, \hat{\vartheta}_\chi^*)_H,$$

which implies that

$$(\hat{y}_k, \hat{\vartheta}_\chi^*)_H = 0 \text{ for } k \ne \chi. \tag{56}$$

Moreover, by (17), we have

$$\int_0^1 \{y_k''^2(x) + q(x) y_k'^2(x)\} dx = \lambda_k \left\{\int_0^1 y_k^2(x) dx + a y_k^2(0) + b y_k'^2(1) - c a y_k^2(1)\right\}. \tag{57}$$

If $\lambda_k \ne 0$, then it follows from (57) that

$$(\hat{y}_k, \hat{\vartheta}_k^*) = [\hat{y}_k, \hat{y}_k]_{\Pi_1} = \int_0^1 y_k^2(x) dx + a y_k^2(0) + b y_k'^2(1) - c a y_k^2(1) \ne 0. \tag{58}$$

If $\lambda_k = 0$, then by Remark 3 we have

$$(\hat{y}_k, \hat{\vartheta}_k^*) = [\hat{y}_k, \hat{y}_k]_{\Pi_1} = 1 + a - c \ne 0. \tag{59}$$

In the case of $c > 1$ and $a = c - 1$ the eigenvalue $\lambda_1 = 0$ is double. Then by Remark 3 from (52) we get

$$(\hat{y}_1, \vartheta_2^*)_H = [\hat{y}_1, \hat{y}_1]_{\Pi_1} = 1 + a - c = 0, \tag{60}$$

and consequently,

$$(\hat{y}_2, \vartheta_1^*)_H = [\hat{y}_2^*, \hat{y}_2^*]_{\Pi_1} + (d + \hat{d})[\hat{y}_1, \hat{y}_2^*], \tag{61}$$

$$(\hat{y}_1, \vartheta_1^*)_H = [\hat{y}_1, \hat{y}_2^*]_{\Pi_1} \text{ and } (\hat{y}_2, \vartheta_2^*)_H = [\hat{y}_1, \hat{y}_2^*]_{\Pi_1}. \tag{62}$$

By following the arguments in Lemma 6. 2 of [1] we can show that

$$[\hat{y}_1, \hat{y}_2^*]_{\Pi_1} \neq 0. \tag{63}$$

Thus by (56), (58), (59) (61)-(63) we have the following results.

Lemma 8. *Let* $\tau_k = [\hat{y}_k, \hat{y}_k]_{\Pi_1}$ *for* $k \geq 3$, *and* $\tau_1 = [\hat{y}_1, \hat{y}_1]_{\Pi_1}$, $\tau_2 = [\hat{y}_2, \hat{y}_2]_{\Pi_1}$ *if* $c \leq 1$ *and* $c > 1$, $a \neq c-1$, $\tau_1 = \tau_2 = [\hat{y}_1, \hat{y}_2^*]_{\Pi_1}$ *if* $c > 1$, $a = c-1$. *Then* $\tau_k \neq 0$ *for any* $k \in \mathrm{N}$.

Lemma 9. *The system* $\{\hat{\vartheta}_k\}_{k=1}^{\infty}$, $\hat{\vartheta}_k = \{\vartheta_k, s_k, t_k^*, r_k\}$, *conjugate to the system* $\{\hat{y}_k\}_{k=1}^{\infty}$ *is determined by*

$$\hat{\vartheta}_k = \tau_k^{-1} \hat{\vartheta}_k^*, \tag{64}$$

where $\hat{d} = -(d + \tau_1^{-1}[\hat{y}_1, \hat{y}_2^*]_{\Pi_1})$.

Let i, j and ν be different arbitrary fixed natural numbers and

$$\Delta_{i,j,\nu} = \begin{vmatrix} s_i & t_i & r_i \\ s_j & t_j & r_j \\ s_\nu & t_\nu & r_\nu \end{vmatrix}. \tag{65}$$

Theorem 3. *Let* i, j *and* ν *be different arbitrary fixed natural numbers. If* $\Delta_{i,j,\nu} \neq 0$, *then the system* $\{y_k\}_{k=1, k\neq i, j, \nu}^{\infty}$ *of root functions of problem (1)-(5) forms a basis in* $L_p(0, 1)$, $1 < p < \infty$, *which is an unconditional basis for* $p = 2$. *If* $\Delta_{i,j,\nu} = 0$, *then the system* $\{y_k\}_{k=1, k\neq i, j, \nu}^{\infty}$ *is not complete and not minimal in* $L_p(0, 1)$, $1 < p < \infty$.

The proof of this theorem is similar to that of [2, Theorem 4.1].

Using the asymptotics of eigenvalues and eigenfunctions of problem (1)-(5), we can establish sufficient conditions for the system to form a basis in $L_p(0, 1)$, $1 < p < \infty$.

By following the arguments in Theorem 5.4 of [4] we can obtain the asymptotic formulas

$$\lambda_k = \left(k - \frac{11}{4}\right)\pi + \frac{q_0 + 2/a - 4/c}{4k\pi} + O\left(\frac{1}{k}\right), \tag{66}$$

$$y_k(0) = 4ibc\rho_k^{11} e^{\rho_k} D_k \left(1 + \frac{q_0 - 4/c}{4\rho_k} + O\left(\frac{1}{\rho_k^2}\right)\right), \tag{67}$$

$$y_k(1) = 4ib\rho_k^{10} e^{\rho_k} D_k \left(1 + \frac{q_0}{4\rho_k} + O\left(\frac{1}{\rho_k^2}\right)\right), \tag{68}$$

$$y_k'(1) = 4ic\rho_k^{9} e^{\rho_k} D_k \left(1 + \frac{q_0 - 4/c}{4\rho_k} + O\left(\frac{1}{\rho_k^2}\right)\right), \tag{69}$$

where $D_k, k \in \mathrm{N}$, is some nonzero constants, $q_0 = \int_0^1 q(x)\,dx$ and $\rho_k = \sqrt[4]{\lambda_k}$, $k \in \mathrm{N}$.

Note that $y_k'(1) \neq 0$ for any $k \geq 3$. Then, we can normalize the function y_k, $k \in \mathrm{N}$, by choosing $y_k'(1) = 1$. Hence it follows from (67) that

$$D_k = \frac{1}{4ic\rho_k^9 e^{\rho_k}} \left(1 - \frac{q_0 - 4/c}{4\rho_k} + O\left(\frac{1}{\rho_k^2}\right)\right),$$

and consequently, by (67)-(69), we get

$$y_k(0) = (-1)^k \frac{b}{a\sqrt{2}} \rho_k \left(1 + +O\left(\frac{1}{\rho_k}\right)\right), \tag{70}$$

$$y_k(1) = \frac{b}{c} \rho_k e^{\rho_k} D_k \left(1 + \frac{1}{c\rho_k} + O\left(\frac{1}{\rho_k^2}\right)\right). \tag{71}$$

Let i, j and ν be different arbitrary fixed natural numbers which are greater than 3. Then in this case we have $s_k = \tau_k^{-1} a y_k(0)$, $t_k = \tau_k^{-1} b y_k'(1)$, $r_k = -\tau_k^{-1} c y_k(1)$, and consequently, by (65) we get

$$\Delta_{i,j,\nu} = \begin{vmatrix} s_i & t_i & r_i \\ s_j & t_j & r_j \\ s_\nu & t_\nu & r_\nu \end{vmatrix} = -abc\tau_i^{-1}\tau_j^{-1}\tau_\nu^{-1} \begin{vmatrix} y_i(0) & y_i'(1) & y_i(1) \\ y_j(0) & y_j'(1) & y_j(1) \\ y_\nu(0) & y_\nu'(1) & y_\nu(1) \end{vmatrix}. \tag{72}$$

Note that $y_2(x) \equiv 1$ and $y_k(0) \neq 0$ for any $k \in \mathrm{N}$.

Let $c > 1$ and $a < c - 1$. Then by Theorem 2 we have

$$0 = \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ \ldots \ .$$

Let $i=1$ and j, ν, $\nu \geq 3$, be arbitrary fixed sufficiently large natural number such that j is even and ν is odd. Then, by (70) and (71), we have

$$\tilde{\Delta}_{1,j,\nu} = \begin{vmatrix} y_1(0) & y_1'(1) & y_1(1) \\ y_j(0) & y_j'(1) & y_j(1) \\ y_\nu(0) & y_\nu'(1) & y_\nu(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ y_j(0) & y_j'(1) & y_j(1) \\ y_\nu(0) & y_\nu'(1) & y_\nu(1) \end{vmatrix} = \frac{b}{a\sqrt{2}} \begin{vmatrix} 1 & 0 & 1 \\ \rho_j & 1 & \frac{b}{c}\rho_j + \frac{b}{c^2} \\ \rho_\nu & 1 & \frac{b}{c}\rho_\nu + \frac{b}{c^2} \end{vmatrix} =$$

$$= \frac{b^2}{ac\sqrt{2}} \begin{vmatrix} 1 & 0 & 1 \\ \rho_j & 1 & \rho_j \\ -\rho_\nu & 1 & \rho_\nu \end{vmatrix} == \frac{b^2}{ac\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 \\ \rho_j & 1 & 0 \\ -\rho_\nu & 1 & 2\rho_\nu \end{vmatrix} = 2\rho_\nu \frac{b^2}{ac\sqrt{2}} \begin{vmatrix} 1 & 0 \\ \rho_j & 1 \end{vmatrix} = 2\rho_\nu \frac{b^2}{ac\sqrt{2}}$$

$$= 2\rho_\nu \frac{b^2}{ac\sqrt{2}}(1-\rho_j) < 0,$$

whence, by (72), implies that

$$\Delta_{i,j,\nu} \neq 0.$$

Thus, by Theorem 3, we have proved the following theorem.

Theorem 4. *Let $c>1$ and $a<c-1$, $i=1$ and j, ν, $\nu \geq 3$, be arbitrary fixed sufficiently large natural number such that j is even and ν is odd. Then the system $\{y_k\}_{k=1, k\neq 1, j, \nu}^{\infty}$ of root functions of problem (1)-(5) forms a basis in $L_p(0, 1)$, $1<p<\infty$, which is an unconditional basis for $p=2$.*

Finally we can look at various cases in a similar way.

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