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MODULUS OF CONTINUITY İN WEAK LEBESGUE SPACES AND ITS PROPERTIES

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Abstract

In the present paper, we introduced the concept of the modulus of continuity of the functions from the weak Lebesgue spaces, studied its properties and found a criterion for convergence to zero of the modulus of continuity of the function from the weak Lebesgue spaces.

Keywords: Modulus of continuity, Lebesgue spaces, weak Lebesgue space, distribution function.
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1. Introduction

Let $L_p(T)$, $1 \le p < \infty$, the space of all measurable 2π-periodic functions with finite $L_p(T)$-norm $\|f\|_p = \left(\frac{1}{\pi} \int_T |f(x)|^p \, dx \right)^{1/p}$, and $L_\infty(T) = C(T)$ the space

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of all continuous 2π-periodic functions with uniform norm $\|f\|_\infty = \max_{x\in T}|f(x)|$, where $T=[-\pi,\pi]$; let $E_n(f)_p$ be the best approximation of a function f in the metric $L_p(T)$ by trigonometric polynomials of order at most n, $n\in Z_+$; and let $\omega(f,\delta)_p = \sup_{0<h\leq\delta}\|f(\cdot+h)-f(\cdot)\|_p$, $\delta\geq 0$.

It was proved by D. Jackson that (see [1]) if $f\in L_p(T)$, $1\leq p\leq\infty$, then

$$E_n(f)_p \leq c\cdot\omega\left(f,\frac{\pi}{n+1}\right)_p, \; n\in Z_+,$$

where c is an absolute constant.

This central theorem gave impetus to the intensive development of approximation theory in the spaces L_p. Further, for the development of the theory of approximation in other function spaces, an analogue of Jackson's theorem in these spaces was obtained (see [2–9] and many references therein).

Weak Lebesgue spaces are function spaces which are closely related to L_p spaces. The weak Lebesgue spaces meets in many areas of mathematics. For example, the conjugate functions of Lebesgue integrable functions belong to the weak Lebesgue space (see [10]). The difficulty of working with the weak Lebesgue spaces is that the weak Lebesgue spaces is not a normed space. Moreover, infinitely differentiable (even continuous) functions are not dense in this spaces. Due to this, the theory of approximation was not produced in this space. In the present paper, we introduced the concept of the modulus of continuity of the functions from the weak Lebesgue spaces, studied its properties and found a criterion for convergence to zero of the modulus of continuity of the function from the weak Lebesgue spaces.

2. Weak Lebesgue spaces

Let (X,μ) be a measure space and f be a measurable function on (X,μ). The distribution function of f is the function D_f defined on $[0,\infty)$ as follows:

$$D_f(\lambda)=\mu(\{x\in X : |f(x)|\geq \lambda\}).$$

It follows from definition that D_f is a decreasing function of λ (not necessarily strictly).

Let (X,μ) be a measure space, f and g be a measurable functions on (X,μ), then the following properties holds:

1) if $|g|\leq |f|$ μ-a.e., then $D_g \leq D_f$;

2) $D_{cf}(\lambda)=D_f\left(\frac{\lambda}{|c|}\right)$ for any $c\in R\setminus\{0\}$ and $\lambda\geq 0$;

3) $D_{f+g}(\lambda_1+\lambda_2)\leq D_f(\lambda_1)+D_g(\lambda_2)$ for any $\lambda_1,\lambda_2\geq 0$;

4) $D_{f\cdot g}(\lambda_1\cdot\lambda_2)\leq D_f(\lambda_1)+D_g(\lambda_2)$ for any $\lambda_1,\lambda_2\geq 0$.

For more details on distribution function see ([11]).

Let (X,μ) be a measurable space and $0<p<\infty$. Let us denote by $L_{(p,\infty)}(X)$ the set of functions f, satisfying the condition

$$\exists C>0 \quad \forall\lambda>0 \quad \mu(\{x\in X : |f(x)|\geq\lambda\})\leq\left(\frac{C}{\lambda}\right)^p.$$

Proposition 1 [12]. Let $f\in L_{(p,\infty)}(X)$ with $0<p<\infty$. Then

$$\inf\left\{C>0 : D_f(\lambda)\leq\left(\frac{C}{\lambda}\right)^p\right\}=\left(\sup_{\lambda>0}\lambda^p D_f(\lambda)\right)^{1/p}=\sup_{\lambda>0}\lambda\left(D_f(\lambda)\right)^{1/p}.$$

Definition 1. For $0<p<\infty$ the set of functions $L_{(p,\infty)}(X)$ with bounded quasi-norm

$$\|f\|_{WL_p(X)}=\inf\left\{C>0 : D_f(\lambda)\leq\left(\frac{C}{\lambda}\right)^p\right\}=\left(\sup_{\lambda>0}\lambda^p D_f(\lambda)\right)^{1/p}=\sup_{\lambda>0}\lambda\left(D_f(\lambda)\right)^{1/p} \tag{1}$$

is called a weak L_p-space and is denoted by $WL_p(X)$.

Note that for any $WL_p(X)$ the inequality

$$\|f+g\|_{WL_p(X)} \le \left(\|f\|_{WL_p(X)}^{\frac{p}{p+1}} + \|g\|_{WL_p(X)}^{\frac{p}{p+1}} \right)^{\frac{p+1}{p}} \le 2^{1/p} \left(\|f\|_{WL_p(X)} + \|g\|_{WL_p(X)} \right) \quad (2)$$

shows that (1) is indeed a quasi-norm.

It follows from Chebyshev inequality that for any $0<p<\infty$

$$L_p(X) \subset WL_p(X)$$

and for any $f \in L_p(X)$

$$\|f\|_{L_p(X)} \le \|f\|_{WL_p(X)}.$$

Modulus of continuity of functions from a weak Lebesgue spaces and its properties

Let $X=[a,b]\subset R$ and $\mu=m$ is Lebesgue measure on $[a,b]$. For any $f\in WL_p([a,b])$, $0<p<\infty$ we put

$$\omega_{\text{weak}}(f;\delta)_p = \sup_{0<h\le\delta} \|f(\cdot+h)-f(\cdot)\|_{WL_p([a,b-h])} =$$

$$= \sup_{0<h\le\delta} \left(\sup_{\lambda>0} \lambda \cdot m\{x\in[a,b-h]: |f(x+h)-f(x)| \ge \lambda\}^{1/p} \right), \; 0<\delta\le b-a,$$

$$\omega^*_{\text{weak}}(f;\delta)_p = \sup_{0<h\le\delta} \|f(\cdot+h)-f(\cdot)\|_{WL_p([a,b])} =$$

$$= \sup_{0<h\le\delta} \left(\sup_{\lambda>0} \lambda \cdot m\{x\in[a,b]: |f(x+h)-f(x)| \ge \lambda\}^{1/p} \right), \; \delta>0,$$

where in the second case the function f is assumed to be extended by periodicity with period $b-a$. The quantities $\omega_{\text{weak}}(f;\delta)_p$ and $\omega^*_{\text{weak}}(f;\delta)_p$ are called modulus of continuity of the function $f\in WL_p([a,b])$ ($\omega^*_{\text{weak}}(f;\delta)_p$ is the periodic modulus of continuity).

We note some properties of the modulus of continuity $\omega_{\text{weak}}(f;\delta)_p$.

Property 1. For every $f\in WL_p([a,b])$, the modulus of continuity

$\omega_{\text{weak}}(f;\delta)_p$ is a nondecreasing function.

Property 2. For every $f \in WL_p([a,b])$ and $0<\delta \le b-a$

$$\omega_{\text{weak}}(f;\delta)_p \le 2^{1+1/p}\|f\|_{WL_p([a,b])}.$$

Property 3. For every $f,g \in WL_p([a,b])$ and $0<\delta \le b-a$

$$\omega_{\text{weak}}(f+g;\delta)_p \le \left(\omega_{\text{weak}}(f;\delta)_p^{\frac{p}{p+1}}+\omega_{\text{weak}}(g;\delta)_p^{\frac{p}{p+1}}\right)^{\frac{p+1}{p}} \le$$

$$\le 2^{1/p}\left(\omega_{\text{weak}}(f;\delta)_p+\omega_{\text{weak}}(g;\delta)_p\right).$$

Property 4. If $f \in WL_p([a,b])$, then for every $\delta_1,\delta_2>0$, $\delta_1+\delta_2 \le b-a$

$$\omega_{\text{weak}}(f;\delta_1+\delta_2)_p \le \left(\omega_{\text{weak}}(f;\delta_1)_p^{\frac{p}{p+1}}+\omega_{\text{weak}}(f;\delta_2)_p^{\frac{p}{p+1}}\right)^{\frac{p+1}{p}} \le$$

$$\le 2^{1/p}\left(\omega_{\text{weak}}(f;\delta_1)_p+\omega_{\text{weak}}(f;\delta_2)_p\right).$$

Property 5. If $f \in WL_p([a,b])$, then for every $k \in N$ and $0<\delta \le \frac{b-a}{k}$

$$\omega_{\text{weak}}(f;k\delta)_p \le k^{1+1/p}\omega_{\text{weak}}(f;\delta)_p.$$

Property 1 is obviously, properties 2, 3, and 4 follow from inequality (2), and property 5 follows from property 4. Indeed, if $k=1$, then property 5 is obviously. If property 5 holds for some $k \in N$, then it follows from property 4 that

$$\omega_{\text{weak}}(f;(k+1)\delta)_p \le \left(\omega_{\text{weak}}(f;k\delta)_p^{\frac{p}{p+1}}+\omega_{\text{weak}}(f;\delta)_p^{\frac{p}{p+1}}\right)^{\frac{p+1}{p}} \le$$

$$\le (k+1)^{1+1/p}\omega_{\text{weak}}(f;\delta)_p,$$

and this means that the property 5 holds for $k+1$. Then it follows from mathematical induction that the property 5 holds for every $k \in N$.

But the equation

$$\lim_{\delta \to 0} \omega_{\text{weak}} \left(f ; \delta \right)_p = 0 \tag{3}$$

overall not satisfied. For example, the function $f(x) = x^{-1/p}$ belongs to the class of functions $WL_p([0,1])$, but for any $\delta > 0$ we have $\omega_{\text{weak}} \left(f ; \delta \right)_p = 1$, and, therefore, equation (3) does not holds for this function.

Theorem 1. The modulus of continuity of the function $f \in WL_p([a,b])$ satisfies equation (3) if and only if

$$\lim_{\lambda \to +\infty} \lambda \cdot m\{x \in [a,b] : |f(x)| \geq \lambda\}^{1/p} = 0 . \tag{4}$$

Proof. Necessity. Let the equation (3) holds. Let us prove that the equation (4) holds. Assume that the equation (4) does not hold. Then

$$\limsup_{\lambda \to +\infty} \lambda \cdot m\{x \in [a,b] : |f(x)| \geq \lambda\}^{1/p} = \alpha > 0 .$$

It follows from here that there is a sequence of positive numbers $\{\lambda_n\}_{n=1}^{\infty}$, such that $\lim\limits_{n \to \infty} \lambda_n = +\infty$ and for every $n \in N$

$$m\{x \in [a,b] : |f(x)| \geq \lambda_n\} > \left(\frac{\alpha}{2\lambda_n} \right)^p . \tag{5}$$

Denote $\varepsilon_0 = \dfrac{\alpha}{8 \cdot 2^{1/p}} > 0$. It follows from (3) that there exists $0 < \delta_0 < \dfrac{b-a}{2}$ such that for every $0 < h \leq \delta_0$ and $\lambda > 0$

$$\lambda \cdot m\{x \in [a, b-h] : |f(x+h) - f(x)| \geq \lambda\}^{1/p} \leq \varepsilon_0 . \tag{6}$$

Denote

$$\Phi_n = \left\{ (x,h) : 0 < h \leq \delta_0 ,\ x \in [a, b-h],\ |f(x+h) - f(x)| \geq \frac{\lambda_n}{2} \right\} .$$

It follows from (6) that for every $0 < h \leq \delta_0$

$$m\left\{ x \in [a, b-h] : |f(x+h) - f(x)| \geq \frac{\lambda_n}{2} \right\} \leq \left(\frac{2\varepsilon_0}{\lambda_n} \right)^p .$$

This implies the estimate

$$m(\Phi_n) \le \left(\frac{2\varepsilon_0}{\lambda_n}\right)^p \cdot \delta_0 = \frac{1}{2} \cdot \left(\frac{\alpha}{4\lambda_n}\right)^p \cdot \delta_0, \tag{7}$$

where $m(\Phi_n)$ denotes the Lebesgue measure of the set Φ_n.

It follows from inclusions

$$\left\{x \in [a, b-h] : |f(x)| \ge \lambda_n \ \wedge \ |f(x+h)| < \frac{\lambda_n}{2}\right\} \subset \left\{x \in [a, b-h] : |f(x+h) - f(x)| \ge \frac{\lambda_n}{2}\right\},$$

$$\left\{x \in [a, b-h] : |f(x+h)| \ge \lambda_n \ \wedge \ |f(x)| < \frac{\lambda_n}{2}\right\} \subset \left\{x \in [a, b-h] : |f(x+h) - f(x)| \ge \frac{\lambda_n}{2}\right\},$$

that

$$2m(\Phi_n) \ge m\left\{(x,h) : 0 < h \le \delta_0,\ x \in [a, b-h],\ |f(x)| \ge \lambda_n \ \wedge \ |f(x+h)| < \frac{\lambda_n}{2}\right\} +$$

$$+ m\left\{(x,h) : 0 < h \le \delta_0,\ x \in [a, b-h],\ |f(x+h)| \ge \lambda_n \ \wedge \ |f(x)| < \frac{\lambda_n}{2}\right\} =$$

$$= m\left\{(x,h) : 0 < h \le \delta_0,\ x \in [a, b-h],\ |f(x)| \ge \lambda_n \ \wedge \ |f(x+h)| < \frac{\lambda_n}{2}\right\} +$$

$$+ m\left\{(x,h) : 0 < h \le \delta_0,\ x \in [a+h, b],\ |f(x)| \ge \lambda_n \ \wedge \ |f(x-h)| < \frac{\lambda_n}{2}\right\} =$$

$$= m\left\{(x,h) : x \in [a, b],\ 0 < h \le \min\{\delta_0, b-x\},\ |f(x)| \ge \lambda_n \ \wedge \ |f(x+h)| < \frac{\lambda_n}{2}\right\} +$$

$$+ m\left\{(x,h) : x \in [a, b],\ 0 < h \le \min\{\delta_0, x-a\},\ |f(x)| \ge \lambda_n \ \wedge \ |f(x-h)| < \frac{\lambda_n}{2}\right\}.$$

Considering that for every $x \in [a, b]$

$$m\left\{h : \ 0 < h \le \min\{\delta_0, b-x\},\ |f(x+h)| < \frac{\lambda_n}{2}\right\} +$$

$$+m\left\{h:\ 0<h\le \min\{\delta_0, x-a\},\quad |f(x-h)|<\frac{\lambda_n}{2}\right\}=$$

$$=\min\{\delta_0, b-x\}-m\left\{h:\ 0<h\le \min\{\delta_0, b-x\},\quad |f(x+h)|\ge\frac{\lambda_n}{2}\right\}+$$

$$\min\{\delta_0, x-a\}-m\left\{h:\ 0<h\le \min\{\delta_0, x-a\},\quad |f(x-h)|\ge\frac{\lambda_n}{2}\right\}\ge$$

$$\ge \min\{\delta_0, b-x\}-\left(\frac{2}{\lambda_n}\|f\|_{WL_p([a,b])}\right)^p+\min\{\delta_0, x-a\}-\left(\frac{2}{\lambda_n}\|f\|_{WL_p([a,b])}\right)^p\ge$$

$$\ge \delta_0-2\cdot\left(\frac{2}{\lambda_n}\|f\|_{WL_p([a,b])}\right)^p,$$

due to inequality (5) we get that

$$m(\Phi_n)\ge\frac{1}{2}\cdot\left(\frac{\alpha}{2\lambda_n}\right)^p\cdot\left(\delta_0-2\cdot\left(\frac{2}{\lambda_n}\|f\|_{WL_p([a,b])}\right)^p\right). \qquad (8)$$

Then it follows from inequalities (7) and (8) that

$$\delta_0-2\cdot\left(\frac{2}{\lambda_n}\|f\|_{WL_p([a,b])}\right)^p\le\frac{\delta_0}{2^p}.$$

But this is impossible due to the condition $\lim\limits_{n\to\infty}\lambda_n=+\infty$. The resulting contradiction proves the validity of equation (4).

Sufficiency. Let the equation (4) be satisfied. Let us prove that equation (3) holds. Let us assume that (3) is not satisfied. Then there exist a number $\varepsilon_0>0$ and sequences of positive numbers $\{h_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty}h_n=+\infty$ and for every $n\in N$

$$\lambda_n\cdot m\{x\in[a,b-h_n]:|f(x+h_n)-f(x)|\ge\lambda_n\}^{1/p}>\varepsilon_0. \qquad (9)$$

It follows from inclusion

$$\{x\in[a,b-h_n]:|f(x+h_n)-f(x)|\ge\lambda_n\}\subset$$

$$\subset\left\{x\in[a,b-h_n]:|f(x)|\geq\frac{\lambda_n}{2}\right\}\cup\left\{x\in[a,b-h_n]:|f(x+h_n)|\geq\frac{\lambda_n}{2}\right\}$$

that

$$m\{x\in[a,b-h_n]:|f(x+h_n)-f(x)|\geq\lambda_n\}\leq 2m\left\{x\in[a,b]:|f(x)|\geq\frac{\lambda_n}{2}\right\}.$$

It follows from here and from (9) that

$$m\left\{x\in[a,b]:|f(x)|\geq\frac{\lambda_n}{2}\right\}>\frac{1}{2}\cdot\left(\frac{\varepsilon_0}{\lambda_n}\right)^p. \tag{10}$$

Inequalities (4) and (10) show that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded. Therefore the sequence $\{\lambda_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$. Let

$$\lambda_0=\lim_{n\to\infty}\lambda_n.$$

It follows from (9) that

$$\lambda_0\geq\frac{\varepsilon_0}{(b-a)^{1/p}}>0.$$

Therefore there exists $k_0\in N$ that for every $k>k_0$

$$\frac{\lambda_0}{2}<\lambda_{n_k}<2\lambda_0.$$

Then from inequality (9) we obtain that for any $k>k_0$

$$m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f\left(x+h_{n_k}\right)-f(x)\right|\geq\frac{\lambda_0}{2}\right\}\geq$$

$$\geq m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f\left(x+h_{n_k}\right)-f(x)\right|\geq\lambda_{n_k}\right\}>\left(\frac{\varepsilon_0}{\lambda_{n_k}}\right)^p>\left(\frac{\varepsilon_0}{2\lambda_0}\right)^p. \tag{11}$$

It follows from (4) that there exist $M_0>0$ such that

$$m\{x\in[a,b]:|f(x)|\geq M_0\}<\frac{1}{4}\left(\frac{\varepsilon_0}{2\lambda_0}\right)^p. \tag{12}$$

Denote

$$f_1(x)=f(x),\ \text{for}\ |f(x)|\leq M_0;\quad f_1(x)=0,\ \text{for}\ |f(x)|>M_0,$$

$$f_2(x)=0,\ \text{for}\ |f(x)|\le M_0;\ \ f_2(x)=f(x),\ \text{for}\ |f(x)|>M_0.$$

Then for every $x\in[a,b]$ we have $f(x)=f_1(x)+f_2(x)$. It follows from the inclusion

$$\left\{x\in\left[a,b-h_{n_k}\right]:\left|f\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{2}\right\}\subset$$

$$\subset\left\{x\in\left[a,b-h_{n_k}\right]:\left|f_1\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{4}\right\}\cup\left\{x\in\left[a,b-h_{n_k}\right]:\left|f\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{4}\right\}$$

and from (12) that

$$m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{2}\right\}\le$$

$$\le m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f_1\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{4}\right\}+\frac{1}{2}\cdot\left(\frac{\varepsilon_0}{2\lambda_0}\right)^p. \qquad (13)$$

Then it follows from (11) and (13) that for every $k>k_0$

$$m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f_1\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{4}\right\}\ge\frac{1}{2}\cdot\left(\frac{\varepsilon_0}{2\lambda_0}\right)^p. \qquad (14)$$

Since the function $f_1(x)$ is bounded, then it is Lebesgue integrable on $[a,b]$. Then it follows from Lebesgue's theorem that

$$\lim_{h\to0+}\int_a^{b-h}\left|f_1(x+h)-f_1(x)\right|dx=0.$$

Therefore,

$$\lim_{k\to\infty}\int_a^{b-h_{n_k}}\left|f_1\left(x+h_{n_k}\right)-f_1(x)\right|dx=0. \qquad (15)$$

On the other hand, it follows from (14) that for every $k>k_0$

$$\int_a^{b-h_{n_k}}\left|f_1\left(x+h_{n_k}\right)-f_1(x)\right|dx\ge$$

$$\ge\frac{\lambda_0}{4}m\left\{x\in\left[a,b-h_{n_k}\right]:\left|f_1\left(x+h_{n_k}\right)-f(x)\right|\ge\frac{\lambda_0}{4}\right\}\ge\frac{\lambda_0}{8}\cdot\left(\frac{\varepsilon_0}{2\lambda_0}\right)^p.$$

But this is impossible due to (15). The resulting contradiction proves the validity of equality (3). This completed the proof of the theorem.

Denote by $WA_p([a,b])$ the class of functions $f \in WL_p([a,b])$ satisfying condition (4). Theorem 1 shows that in the class of functions $WA_p([a,b])$ the modulus of continuity $\omega_{\mathrm{weak}}(f;\delta)_p$ satisfies condition (3).

Note that properties 1-5 and theorem 1 also holds for the periodic modulus of continuity $\omega^*_{\mathrm{weak}}(f;\delta)_p$.

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