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Analytical solution of the Dirac equation for the linear combination of the Manning-Rosen and Yukawa potential in the case of exact spin symmetry

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Abstract

In this paper, the analytically bound state solution of the Dirac equation is obtained for the linear combination of the Manning-Rosen and Yukawa potentials by using Nikiforov-Uvarov method. To overcome the difficulties arising in the case for arbitrary k in the centrifugal part of the Manning-Rosen potential plus the Yukawa potential for bound states, we applied the developed approximation. Analytical expressions for the energy eigenvalue and the corresponding spinor wave functions for an arbitrary value k spin-orbit, radial n and l orbital quantum numbers are obtained. The relativistic energy eigenvalues and corresponding spinor wave functions have been obtained for cases exact spin and pseudospin symmetries by using the Nikiforov-Uvarov method. Furthermore, the corresponding normalized eigenfunctions have been represented as a recursion relation in terms of the Jacobi polynomials for arbitrary k states. A closed form of the normalization constant of the wave functions is also found. It is shown that the energy eigenvalues and eigenfunctions are very sensitive to k spin-orbital quantum number.

Keywords: *Manning-Rosen potential, Yukawa potential, spin symmetry, Nikiforov-Uvarov Method*
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1. Introduction

Since its inception, the main goal of quantum mechanics has been the analytical solution of wave equations for specific potentials of physical interest. It is possible

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to obtain quite perfect information about quantum systems from the wave functions found by solving the wave equations. That is, the wave functions or eigenfunctions of the system contain all the information about the quantum mechanical system [1-4]. Therefore, finding an analytical and bound solution of the Dirac equation is a very important research task. In relativistic quantum mechanics, the Dirac equation is used to describe the dynamics of systems with spin-1/2 [1-3].

The Dirac equation is widely used to investigate physical phenomena in various processes, especially in nuclear and hadron physics. In the study of these fields, it is assumed that the Dirac Hamiltonian has two types of symmetry: exact spin and pseudospin symmetry [5-8]. The cases of spin and pseudospin symmetry in the Dirac equation were first proposed in 1969 by Arima, Hecht, Adler et al. [9-10].

The exact spin and pseudospin symmetries are derived from symmetry $SU(2)$ of the Dirac Hamiltonian from the specific relations between vector $V(r)$ and scalar potentials $S(r)$, that is, from the condition that the difference of vector $V(r)$ and scalar potential $S(r)$ is equal to $\Delta(r) = V(r) - S(r)$ and the sum is equal to $\Sigma(r) = V(r) + S(r)$ constant. The Dirac equation directly includes the differentials of the difference $\Delta(r)$ and the sum $\sum(r)$ of the potentials, that is, the terms $d\Delta(r)/dr$ and $d\sum(r)/dr$ are included. Since $\Delta(r) = const$ is constant, $d\Delta(r)/dr = 0$ is obtained from here, so that $\Delta(r) = const$. In the second case, since $\Sigma(r) = const$ is $d\Sigma/dr = 0$.

The symmetry corresponding to this case is called pseudospin symmetry. The exact spin symmetry creates two twisted states corresponding to these $(n, l, j = l \mp s)$ quantum numbers, which can be viewed as a spin doublet. Here, n -radial, l -orbital and j -integer moment quantum numbers, s -spin quantum number. In the case of pseudo-spin symmetry, there is a quasi-entanglement, which corresponds to two states of the orbital quantum number that differ by a whole unit, i.e. $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$. We can also view these states as pseudospin doublets with quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$.

The pseudospin doublet is characterized by $(\tilde{n} = n - 1, \tilde{l} = l + 1, \tilde{j} = \tilde{l} \pm \tilde{s})$ quantum numbers. Here $\tilde{n}$-pseudo radial $\tilde{n} = n - 1$, $\tilde{l}$-pseudo orbital $\tilde{l} = l + 1$ and $\tilde{s}$-pseudospin $\tilde{s} = 1/2$ quantum numbers are called [11, 12]. It is possible to interpret the pseudo orbital quantum number as the lower component of the Dirac spinor [7]. Many studies have been done to study these two symmetries. For example, to explain the antinucleon spectrum of the nucleus [8-11], the process of nuclear deformation [12], nuclear superdeformation [13], the effective envelope model of the nucleus [14] and also small spin-orbital expansion in hadrons. On the other hand, the effect of the tensor interaction potential on both symmetries shows that all doublets lose their distortion[15]. To investigate this property, the Dirac equation was solved analytically for many potentials taking into account the tensor interaction [16-19].

2. Research method

Taking into account $V(r)$ vector repulsion and $S(r)$ attraction fields of the sum of Manning-Rosen and Yukawa potentials, we can write the Dirac equation M for a mass particle in the following form in the system of ($\hbar = c = 1$) atomic units [1]:

$$[\vec{\alpha}\vec{p} + \beta(M + S(r))]\psi(r) = [E - V(r)]\psi(r). \tag{1}$$

Here they describe the fields of $V(r)$ vector repulsion and $S(r)$ scalar attraction. α and β are the four-dimensional Dirac matrices, and E is the relativistic energy of the system. The spherically symmetric dirac spinor wave function can be written in the following form, i.e

$$\psi_{nk} = \frac{1}{r}\begin{bmatrix} F_{nk}(r) & Y_{jm}^{l}(\theta,\varphi) \\ iG_{nk}(r) & Y_{im}^{\tilde{l}}(\theta,\varphi) \end{bmatrix} \tag{2}$$

$F_{nk}(r)$ and $G_{nk}(r)$ are real quadratic integrable functions, $m$is the magnetic quantum number, which characterizes the projection of the angular momentum along the z-axis. $Y_{jm}^{l}(\theta,\varphi)$ $Y_{jm}^{\tilde{l}}(\theta,\varphi)$ functions are spherical harmonic functions. At the same time, they satisfy conditions $l(l+1) = k(k+1)$ and $\tilde{l}(\tilde{l}+1) = k(k-1)$. Since the potential we are looking at is spherically symmetric and independent of time, we will solve the stationary Dirac equation.

In order to solve the Dirac equation for the sum of the Manning-Rosen and Yukawa potentials, if we consider the function (2) in equation (1), then we get two related equations:

$$\left(\frac{d}{dr} + \frac{k}{r}\right)F_{nk}(r) = [M + E_{nk} + S(r) - V(r)]G_{nk}(r), \tag{3}$$

$$\left(\frac{d}{dr} - \frac{k}{r}\right)G_{nk}(r) = [M - E_{nk} + S(r) + V(r)]F_{nk}(r). \tag{4}$$

From here, if we find function $G_{nk}(r)$ from equation (3) and find function $F_{nk}(r)$ from equation (4) and consider it in equation (3), then we get second order differential equations for these functions in the following form:

$$\left\{\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \sum(r)] + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{k}{r}\right)}{M + E_{nk} - \Delta(r)}\right\} \times$$

$$\times F_{nk}(r) = 0, \tag{5}$$

$$\left\{\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \sum(r)] - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{k}{r}\right)}{M - E_{nk} + \Sigma(r)}\right\} \times$$

$$\times G_{nk}(r) = 0. \tag{6}$$

Here they are designated as $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$. Also, when there are $k = \pm 1, \pm 2, \pm 3, \ldots.$ they are designated as $j = |k| - 1/2, l = |k + 1/2| - 1/2, \tilde{l} = |k - 1/2| - 1/2$ and $l(l+1) = k(k+1), \tilde{l}(\tilde{l}+1) = k(k-1)$. At the same time, Dirac spinors for bound states also satisfy conditions $F_{nk}(0) = G_{nk}(0) = 0$ and $F_{nk}(\infty) = G_{nk}(\infty) = 0$.

The Manning-Rosen potential is widely used in mathematical modeling of oscillations and vibrations of diatomic molecules. It is also used in the construction of suitable models for the mathematical description of other physical phenomena, as follows [20, 21]:

$$V_{MR}(r) = \frac{\hbar^2}{2\mu\delta^2}\left[\frac{\alpha(\alpha-1)e^{-2r/\delta}}{(1-e^{-r/\delta})^2} - \frac{Ae^{-r/\delta}}{1-e^{-r/\delta}}\right]. \tag{7}$$

Yukawa potential, considered as an effective potential to describe the strong interaction between nucleons [22]

$$V_Y(r) = -\frac{V_0 e^{-\delta r}}{r}, \tag{8}$$

where A and α are dimensionless constants, and δ is a screening parameter. V_0 determines the strength of the interaction.

We can write the linear sum of the Manning-Rosen and Yukawa potentials as follows:

$$V_{MRY}(r) = \frac{2\delta^2\alpha(\alpha-1)e^{-4\delta r}}{M(1-e^{-2\delta r})^2} - \frac{2\delta^2}{M}\cdot\frac{Ae^{-2\delta r}}{1-e^{-2\delta r}} - \frac{2\delta V_0 e^{-2\delta r}}{1-e^{-2\delta r}} =$$

$$= \frac{V_{01}e^{-4\delta r}}{(1-e^{-2\delta r})^2} - \frac{V_{023}e^{-2\delta r}}{1-e^{-2\delta r}} \tag{9}$$

Here

$$V_{01} = \frac{2\delta^2\alpha(\alpha-1)}{M} \text{ and } V_{023} = V_{02} + V_{03} = \frac{2\delta^2 A}{M} + 2\delta V_0. \tag{10}$$

In case of exact spin symmetry $d\Delta(r)/dr = 0$, since $\Delta(r) = const.$ [18, 19]. $\Sigma(r)$ is equal to the sum of the Manning-Rosen and Yukawa potentials, i.e $\Sigma(r) = V_{MRY}(r)$.

If we consider the potential (9) in equation (5), then we get it for exact spin symmetry:

$$\left\{\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - [M + E_{nk} - C]\Sigma(r) + [E_{nk}^2 - M^2 + C(M - E_{nk})]\right\}F_{nk}(r) = 0 \tag{11}$$

It can be seen from the equation (10) that it cannot be solved analytically when $k=-1$and $k=0$ are present. Because, in this case, the spin-orbit coupling parameter $k(k+1)/r^2$ is equal to zero. To overcome this problem, let us apply the following approximation to the centrifugal potential in equation (11). It is possible to apply this approximation if it satisfies $\delta r << 1$ criteria [23-25]:

$$\frac{1}{r^2}=\frac{4\delta^2 e^{-2\delta r}}{(1-e^{-2\delta r})^2}. \quad (12)$$

Then we will buy:

$$\left[\frac{d^2}{dr^2}-[M+E_{nk}-C]\Sigma(r)+[E_{nk}^2-M^2+C(M-E_{nk})]-\frac{4k(k+1)\delta^2 e^{-2\delta r}}{(1-e^{-2\delta r})^2}\right]\times$$

$$\times F_{nk}(r)=0 \quad (13)$$

Let's introduce the variable $s=e^{-2\delta r}$ to solve this equation. Then we can write equation (13) in the following way:

$$\left[4\delta^2 s^2\frac{d^2}{ds^2}+4\delta^2 s\frac{d}{ds}+[M+E_{nk}-C]\left[\frac{V_{01}e^{-4\delta r}}{(1-e^{-2\delta r})^2}-\frac{V_{023}e^{-2\delta r}}{1-e^{-2\delta r}}\right]+\right.$$

$$\left.+\left[E_{nk}^2-M^2+C(M-E_{nk})\right]-\frac{4k(k+1)\delta^2 e^{-2\delta r}}{(1-e^{-2\delta r})^2}\right]F_{nk}(r)=0 \quad (14)$$

After some simplifications, we can write equation (14) as follows:

$$\frac{d^2F_{nk}}{ds^2}+\frac{1}{s}\frac{dF_{nk}}{ds}+\left[[M+E_{nk}-C]\left[\frac{V_{01}}{4\delta^2(1-s)^2}-\frac{V_{023}}{4\delta^2(1-s)s}\right]+\right.$$

$$\left.+\frac{1}{4\delta^2 s^2}\left[E_{nk}^2-M^2+C(M-E_{nk})\right]-\frac{k(k+1)}{s(1-s)^2}\right]F_{nk}=0. \quad (15)$$

New parameters can be introduced to simplify equation (15):

$$\alpha^2=\frac{V_{01}(M+E_{nk}-C)}{4\delta^2},\beta^2=\frac{V_{023}(M+E_{nk}-C)}{4\delta^2},$$

$$\gamma^2=\frac{M^2-E_{nk}^2-C(M-E_{nk})}{4\delta^2}. \quad (16)$$

Then we get the following equation:

$$\frac{d^2F_{nk}}{ds^2}+\frac{1-s}{s(1-s)}\frac{dF_{nk}}{ds}+\frac{1}{s^2(1-s)^2}\times$$

$$\times[\alpha^2s^2-\beta^2s(1-s)-\gamma^2(1-s)^2-k(k+1)s]F_{nk}=0 \quad (17)$$

(17) equation

$$\chi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\chi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\chi(s)=0 \tag{18}$$

if we compare it with the hypergeometric equation, then for coefficients $\bar{\tau}(s), \sigma(s)$ and $\bar{\sigma}(s)$ we get [26]:

$$\bar{\tau}(s)=1-s, \sigma(s)=s(1-s)$$

and (19)

$$\bar{\sigma}(s)=\alpha^2 s^2-\beta^2 s(1-s)-\gamma^2(1-s)^2-k(k+1)s$$

If we use expressions (18), we can calculate the function:

$$\pi(s)=\frac{\sigma'(s)-\bar{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tau(s)}{2}\right)^2-\bar{\sigma}(s)+k'\sigma(s)}=-\frac{s}{2}\pm$$

$$\pm\sqrt{\frac{s^2}{4}-\alpha^2 s^2+\beta^2 s(1-s)-\gamma^2(1-s)^2+k(k+1)s+k's-k's^2}=-\frac{s}{2}\pm$$

$$\pm\sqrt{\frac{s^2}{4}-\alpha^2 s^2+\beta^2 s-\beta^2 s^2-\gamma^2+2\gamma^2 s-\gamma^2 s^2+k(k+1)s-k's^2+k's}=$$

$$=-\frac{s}{2}\pm\sqrt{\frac{s^2}{4}-(\alpha^2+\beta^2+\gamma^2+k')s^2+(\beta^2+2\gamma^2+k(k+1)+k')s-\gamma^2}=$$

$$=-\frac{s}{2}\pm\sqrt{\frac{1}{4}(1-4(\alpha^2+\beta^2+\gamma^2+k')s^2+(\beta^2+2\gamma^2+k(k+1)+k')s-\gamma^2}$$

$$=-\frac{s}{2}\pm\frac{1}{2}\sqrt{(1+4(\alpha^2+\beta^2-\gamma^2-k')s^2-4(-\beta^2+2\gamma^2-k(k+1)-k')s+4\gamma^2}=$$

$$=-\frac{s}{2}+\frac{1}{2}\sqrt{(4a^2-4k')s^2-4(b-k')+4c}=$$

$$=-\frac{s}{2}\pm\sqrt{(a-k')s^2-(b-k')s+c}, \tag{20}$$

here

$$a=1/4+(\alpha^2+\beta^2+\gamma^2), \tag{21}$$

$$b=+2\gamma^2+\beta^2-k(k+1), \qquad c=+\gamma^2.$$

To find the parameter k' in the expression (19), if we first find the discriminant of the second-order polynomial under the root and make it equal to zero, then we get a quadratic equation with respect to k', i.e.

$$D = (b - k')^2 - 4c(a - k') = k'^2 - (4c - 2b)k' + (b^2 - 4ac).$$

From condition $D = 0$

$$k'^2 - (4c - 2b)k' + (b^2 - 4ac) = 0, \quad (22)$$

we will buy If we solve equation (22) with respect to k', then we get:

$$k'_{1,2} = (2b - 4c)/2 \pm \sqrt{(4c - 2b)^2 - 4(b^2 - 4ac)}/2, \quad (23)$$

$$k'_1 = (b - 2c) + 2\sqrt{c^2 + c(a - b)}, \quad (24)$$

$$k'_2 = (b - 2c) - 2\sqrt{c^2 + c(a - b)}, \quad (25)$$

on the other hand

$$(a - k')s^2 - (b - k)s + c = \left(a - b + 2c - 2\sqrt{c^2 + c(a - b)}\right)s^2 -$$

$$-\left(2c - 2\sqrt{c^2 + c(a - b)}\right)s - c = (As - B)^2, \quad (26)$$

$$A^2 = a - b + 2c - 2\sqrt{c^2 + c(a - b)}.$$

From here we get that

$$2AB = 2c - 2\sqrt{c^2 + c(a - b)},$$

$$B^2 = c, \quad (27)$$

$$A = \sqrt{c} - \sqrt{c + a - b}.$$

$$\sqrt{c + a - b} = \sqrt{+\gamma^2 + 1/4 + \alpha^2 - \beta^2 + \gamma^2 - 2\gamma^2 + \beta^2 + k(k + 1)} =$$

$$= \sqrt{1/4 + k(k + 1) - \alpha^2} = \sqrt{(k + 1/2)^2 + \alpha^2}. \quad (28)$$

Using expressions (24) and (25), we can find four possible functions for the function $\pi(s)$:

$$\pi(s) = -s/2 \pm$$

$$\pm \begin{cases} \left(\sqrt{c} - \sqrt{c + a - b}\right)s - \sqrt{c}, k' = (b - 2c) + 2\sqrt{c^2 + c(a - b)} \\ \left(\sqrt{c} + \sqrt{c + a - b}\right)s - \sqrt{c}, k' = (b - 2c) - 2\sqrt{c^2 + c(a - b)} \end{cases} \quad (29)$$

According to the Nikiforov-Uvarov method, we choose one of the four possible forms of the polynomial $\pi(s)$ such that the derivative of the function $\tau(s)$ for this form polynomial is negative and the root is located in the interval $(0,1)$.

The specific value of energy is found directly from the expression of parameter λ, that is, it is found from the condition that the two equivalent expressions of parameter λ are equal to each other:

$$\lambda = (b - 2c) - 2\sqrt{c^2 + c(a - b)} - \frac{1}{2} - \left[\sqrt{c} + \sqrt{c + a - b}\right]. \qquad (29)$$

For given non-negative integers n, an equation of hypergeometric type has only and only a unique polynomial solution of degree n:

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad (n = 0, 1, 2, \dots) \qquad (29)$$

Here

$$\tau(s) = 1 - \left(2 + 2\left(\sqrt{c} + \sqrt{c + a - b}\right)\right)s + 2\sqrt{c},$$

$$\tau'(s) = -(2 + 2(\sqrt{c} + \sqrt{c + a - b})),$$

then

$$\lambda_n = n\left[2 + 2(\sqrt{c} + \sqrt{c + a - b})\right] - \frac{n(n-1)}{2} \cdot (-2) =$$

$$= 2n\left[1 + (\sqrt{c} + \sqrt{c + a - b})\right] + n(n-1). \qquad (30)$$

We get from the equivalence of expressions (29) and (30):

$$(b - 2c) - 2\sqrt{c} \cdot \sqrt{c + a - b} - 1/2 - \left[\sqrt{c} + \sqrt{c + a - b}\right] =$$

$$= 2n\left[1 + \left(\sqrt{c} + \sqrt{c + a - b}\right)\right] + n(n-1),$$

$$(b - 2c) - 2\sqrt{c} \cdot \sqrt{c + a - b} - 1/2 - \sqrt{c + a - b} =$$

$$= 2n\left[1 + \left(\sqrt{s} + \sqrt{c + a - b}\right)\right] + n(n-1),$$

$$\sqrt{s}\left(2n + 1 + 2\sqrt{c + a - b}\right) =$$

$$= -\beta^2 - k(k+1) - 1/2 - \sqrt{(k + 1/2)^2 + \alpha^2} - n - n^2 - 2n\sqrt{(k + 1/2)^2 + \alpha^2}$$

$$\sqrt{c}\left(2n + 1 + 2\sqrt{(k + 1/2)^2 + \alpha^2}\right) =$$

$$= -\beta^2 - k(k+1) - 1/2 - n(n+1) - (2n+1)\sqrt{(k + 1/2)^2 + \alpha^2}$$

$$\sqrt{c} = \frac{-\beta^2 - k(k+1) - \frac{1}{2} + n(n+1) - (2n+1)\sqrt{(k + 1/2)^2 + \alpha^2}}{2n + 1 + 2\sqrt{\left(k + \frac{1}{2}\right)^2 - \alpha^2}}. \qquad (31)$$

From here we get the following analytical expression for the specific value of energy:

$$c=\left[\frac{\beta^2-k(k+1)-\frac{1}{2}-n(n+1)-(2n+1)\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}{2n+1+2\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\right]^2, \quad (32)$$

$$\gamma^2=\left[\frac{\beta^2-k(k+1)-\frac{1}{2}-n(n+1)-(2n+1)\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}{2n+1+2\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\right]^2. \quad (33)$$

If we consider the expression of γ^2 from expression (15) in (33), then we can write the energy spectrum for the case of exact spin symmetry as follows:

$$M^2-E_{nk}^2-C(M-E_{nk})=$$

$$=\left[\frac{\beta^2-k(k+1)-\frac{1}{2}-n(n+1)-(2n+1)\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}{n+\frac{1}{2}+\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\cdot\delta\right]^2 \quad (34)$$

Here, the Dirac equation for the linear sum of the Manning-Rosen and Yukawa potentials is solved analytically by applying the Nikiforov-Uvarov method in ordinary quantum mechanics for the case of exact spin symmetry. For arbitrary values of spin-orbital, radial and orbital quantum numbers, an analytical expression describing the energy spectrum of the particle was obtained. It is shown that the energy spectrum strongly depends on these quantum numbers.

Now, to find the eigenfunction of a relativistic particle of mass M moving in a Manning-Rosen plus Yukawa potential field by applying the Nikifarov-Uvarov method, let us factorize the function $F_{nk}(s)$ as follows:

$$F_{nk}(s)=\varphi(s)y(s). \quad (35)$$

Let us find the function $\varphi(s)$ by requiring that it satisfies the following condition:

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)}. \quad (36)$$

From here we get for the function $\varphi(s)$:

$$\int\frac{d\varphi(s)}{\varphi(s)}=\int\frac{\pi(s)}{\sigma(s)}ds. \quad (37)$$

$$\frac{\pi(s)}{\sigma(s)}=\frac{\gamma}{s}-\frac{1/2-\sqrt{(k+1/2)^2+\alpha^2}}{1-s},$$

$$\int \frac{d\varphi(s)}{\varphi(s)} = \int \frac{\gamma}{s} ds - \int \frac{\frac{1}{2} - \sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}}{1 - s} ds$$

$$= ln\, s^{\gamma} + ln(1 - s)^{\frac{1}{2} - \sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}},$$

$$ln\, \varphi(s) = ln\left[s^{\gamma} \cdot (1 - s)^{\frac{1}{2} - \sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}}\right],$$

$$\varphi(s) = s^{\gamma} \cdot (1 - s)^{\frac{1}{2} - \sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}}. \quad (38)$$

In order to find the function $y(s)$ in the function (35), it is necessary to find the function $\rho(s)$ first. $\rho(s)$ weight functions are found from solving the Pearson differential equation. The Pearson differential equation is defined as follows:

$$(\sigma\rho)' = \tau\rho. \quad (39)$$

Let's solve this equation:

$$(\sigma' - \tau)\rho = -\sigma\rho',$$

$$\int \frac{d\rho}{\rho} = \int \frac{\tau - \sigma'}{\sigma} ds,$$

$$\tau - \sigma' = 2\sqrt{c}(1 - s) - 2s\sqrt{c + a - b},$$

$$\int \frac{d\rho}{\rho} = \int \frac{2\sqrt{c}}{s} ds - \int \frac{2s\sqrt{c + a - b}}{1 - s} ds = 2\sqrt{c}\, ln\, s + 2\sqrt{c + a - b}\, ln(1 - s),$$

$$ln\, \rho = ln\left(s^{2\sqrt{c}} \cdot (1 - s)^{2\sqrt{c + a - b}}\right),$$

$$\rho = s^{2\sqrt{c}} \cdot (1 - s)^{2\sqrt{c + a - b}} = s^{2\gamma} \cdot (1 - s)^{2\sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}}. \quad (40)$$

The function $y_n(s)$, which is a component of the spinor function $F_{nk}(s)$, is determined by the Rodriques formula in the following way:

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)]. \quad (41)$$

If we consider the expressions of $\sigma(s)$ and $\rho(s)$ in (41), then we get $y_n(s)$ for the function:

$$y_n(s) = \frac{C_n}{s^{2\gamma} \cdot (1 - s)^{2\sqrt{(k + 1/2)^2 + \alpha^2}}} \cdot \frac{d^n}{ds^n}\left[s^{n + 2\gamma} \cdot (1 - s)^{n + 2\sqrt{\left(k + \frac{1}{2}\right)^2 + \alpha^2}}\right]. \quad (42)$$

of Jacobi polynomial [25]

$$P_n^{(a,b)}(s) = \frac{(-1)^n}{n!\,2^n(1-s)^a(1+s)^b}\frac{d^n}{ds^n}[(1-s)^{a+n}\cdot(1+s)^{b+n}]$$

and

$$P_n^{(a,b)}(1-2s) = \frac{C_n}{s^a(1-s)^b}\frac{d^n}{ds^n}[s^{a+n}\cdot(1-s)^{b+n}],$$

properties and using the following expression:

$$\frac{d^n}{ds^n}[s^{a+n}\cdot(1-s)^{b+n}] = C_n s^a(1-s)^b P_n^{(a,b)}(1-2s). \tag{43}$$

then we can write the function (42) in the following way:

$$y_n(s) = C_n P_n^{\left(2\gamma,2\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}\right)}(1-2s). \tag{44}$$

So we can write the function $F_{nk}(s)$ as follows:

$$F_{nk}(s) = \varphi(s)\cdot y(s) = s^\gamma(1-s)^{\frac{1}{2}-\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\times$$

$$\times\frac{C_n}{s^{2\gamma}\cdot(1-s)^{2\sqrt{(k+1/2)^2+\alpha^2}}}\frac{d^n}{ds^n}\left[s^{n+2\gamma}\cdot(1-s)^{n+2\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\right]$$

$$= C_n s^\gamma(1-s)^{\frac{1}{2}-\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}}\cdot P_n^{\left(2\gamma,2\sqrt{\left(k+\frac{1}{2}\right)^2+\alpha^2}\right)}(1-2s). \tag{45}$$

The normalization constant C_n is found from the normalization condition and is as follows:

$$C_n = \sqrt{\frac{2\delta\cdot n!\,(n+k+1+\gamma)\Gamma(2\gamma+1)\Gamma(n+2\gamma+k}{(n+k+1)\Gamma(2\gamma)\Gamma(n+2\gamma+1)\Gamma(n+2k+2)}}. \tag{46}$$

3. Conclusions

The Dirac equation for the linear combination of the Manning-Rosen and Yukawa potentials was solved analytically by applying the Nikiforov-Uvarov method in the cases of exact spin and pseudospin symmetry in ordinary quantum mechanics. Analytical expressions were found for the eigenvalue of the energy of a mass M relativistic particle moving in the field composed of the sum of the Manning-Rosen and Yukawa potentials, eigenfunctions expressed by the Jacobi polynomial. It is shown that in both cases the energy spectrum and eigenfunctions are sensitive to the choice of spin-orbital and radial quantum numbers. The obtained results can be used in the construction of new models of elementary particles, in the study of the spin balance of protons and neutrons.

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