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NONEXISTENCE OF GLOBAL SOLUTIONS IN AN INFINITE DOMAIN FOR A SYSTEM OF THREE SEMILINEAR SECOND-ORDER PARABOLIC EQUATIONS WITH SINGULAR POTENTIAL

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Abstract

In the domain $Q_R' = \{x: |x| > R\} \times (0;+\infty)$ we consider the following problem:

$$\frac{\partial u_i}{\partial t} = \Delta u_i + \frac{C_i}{|x|^2} u_i + |x|^{\sigma_i} |u_{i+1}|^{q_i}, (x,t) \in Q_R'$$

$$u_i\big|_{t=0} = u_{0i}(x), \ \ x \in B_R',$$

where $n > 2, \ C_i < \left(\frac{n-2}{2}\right)^2, \ \sigma_i \in R, \ q_i > 1, u_{0i}(x) \geq 0, \ i = 1,2,3.$ A sufficient condition on the absence of global solutions is obtained. The proof is based on the method of test functions.

Keywords: system of semilinear parabolic equation, absence of global solutions, critical exponent, singular potential

Mathematics Subject Classification *(2020):* 35B33, 35K57, 35K55, 35B25

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1. Introduction

Let $R > 0, \ B_R = \{x: |x| < R\}, \ \ B_R' = \{x: |x| > R\}, \ \ \partial B_R = \{x: |x| = R\},$

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$$B_{R_1,R_2}=\{x: R_1<|x|<R_2\},\ \overline{B}'_R=R^n\setminus B_R,\ Q_R=B_R\times(0;+\infty),$$
$$Q'_R=B'_R\times(0;+\infty),\ \overline{Q}'_R=\overline{B}'_R\times[0;+\infty),\ x=(x_1,\dots,x_n)\in R^n,$$
$$r=|x|=\sqrt{x_1^2+\dots+x_n^2}.$$

In the domain Q'_R we consider the following system of equations:

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=\Delta u_1+\dfrac{C_1}{|x|^2}u_1+|x|^{\sigma_1}|u_2|^{q_1}\\ \dfrac{\partial u_2}{\partial t}=\Delta u_2+\dfrac{C_2}{|x|^2}u_2+|x|^{\sigma_2}|u_3|^{q_2}\\ \dfrac{\partial u_3}{\partial t}=\Delta u_3+\dfrac{C_3}{|x|^2}u_3+|x|^{\sigma_3}|u_1|^{q_3},\end{cases}\tag{0.1}$$

where $n>2,\ C_i<\left(\dfrac{n-2}{2}\right)^2,\ \sigma_i\in R,\ q_i>1,\ i=1,2,3.$

We will write the system briefly as follows:

$$\frac{\partial u_i}{\partial t}=\Delta u_i+\frac{C_i}{|x|^2}u_i+|x|^{\sigma_i}|u_{i+1}|^{q_i},$$

$u_4=u_1.$

We will investigate the issue of the absence of global solutions of the system (0.1), satisfying the initial condition

$$u_i\big|_{t=0}=u_{0i}(x),\ x\in B'_R,\tag{0.2}$$

$u_{0i}(x)\geq 0,\ u_{0i}\in L_{\infty,loc}\in(\overline{B'_R})$.

A solution of problem (0.1), (0.2) is understood in the weak sense, namely the system of functions $u_i(x,t)\in L_{\infty,loc}(\overline{Q'_R})$ is called a weak solution of problem (0.1),(0.2), if

$$\iint\limits_{Q'_R}|x|^{\sigma_i}|u_{i+1}|^{q_i}\eta\,dxdt=-\iint\limits_{Q'_R}u_i\frac{\partial\eta}{\partial t}dxdt-\iint\limits_{Q'_R}u_i\left[\Delta\eta+\frac{C_i}{|x|^2}\eta\right]dxdt-$$
$$\int\limits_{B'_R}u_{0i}(x)\eta(x,0)dx+\int\limits_0^\infty\int\limits_{\partial B'_R}u_i\frac{\partial\eta}{\partial\nu}dsdt$$

for any $\eta(x,t)\in C^{2,1}(\overline{Q}_R'),\ \ \eta\big|_{\partial B_R}=0,\ \ \dfrac{\partial\eta}{\partial\nu}\bigg|_{\partial B_R}\le 0,$ and there exist $R_1>R,$ $T_0>0$ such that $\eta(x,t)\equiv 0$ when $|x|>R_1$ and $t>T_0$, where ν is the unit vector of the outer normal to ∂B_R.

The problems of the absence of global solutions for various classes of differential equations and inequalities play an important role in theory and applications. Therefore, they are under constant attention from mathematicians, and many works have been dedicated to them. An overview of such results can be found in the monograph [1] and in the survey paper [2].

In 1966, Fujita, in his famous work [3], studied the following Cauchy problem for the heat equation with an internal source:

$$\frac{\partial u}{\partial t}=\Delta u+u^q,\qquad (x,t)\in R^n\times(0,+\infty),$$

$$u\big|_{t=0}=u_0(x)\ge 0,\qquad x\in R^n,$$

where $q>1,\ u_0(x)$ is a continuous and bounded function, and he proved that when $1<q<q_{cr}=1+\dfrac{2}{n}$, the stated problem has no non-negative nontrivial global solution, whereas when $q>q_{cr}=1+\dfrac{2}{n}$ it has both a non-negative nontrivial global solution (for small initial data) and a nonnegative local solution (for large initial data). Later, Hayakawa [4] and Kobayashi [5] showed that in the case $q=1+\dfrac{2}{n}$, global solutions also do not exist. Thus, there exists a value of the non-linearity such that, depending on this value, a global solution may or may not exist. This value is called the critical exponent of the non-linearity in the mathematical literature, and the corresponding results are known as Fujita-type theorems. At present, many results have been achieved in the study of Fujita-type theorems. Numerous analogous theorems have been established for various types of nonlinear equations and systems of equations, different types of domains, and various types of boundary conditions. In addition, the influence of many factors—such as spatial dimension, domain shape, diffusion term, convection term, source term, and boundary term—on the Fujita critical exponent has been characterized [6–11]. In the case of a single equation, the Fujita critical exponent is usually a constant, whereas in the case of coupled systems of equations, the Fujita critical exponent is typically a curve. In [12], M. Escobedo and M. A. Herrero were the first to consider the following initial value problem for the system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + |x|^{\sigma_1} v^{q_1} \\ \dfrac{\partial v}{\partial t} = \Delta v + |x|^{\sigma_2} u^{q_2} \end{cases} \qquad (x,t) \in R^n \times (0,+\infty), \tag{0.3}$$

$$u\big|_{t=0} = u_0(x), v\big|_{t=0} = v_0(x), \qquad x \in R^n, \tag{0.4}$$

where $q_1 > 0, q_2 > 0$, $u_0(x), v_0(x)$ continuous, bounded, and non-negative functions. It was proved that for $\sigma_1 = \sigma_2 = 0$ the critical curve of this problem has the form $(pq)_{cr} = 1 + \frac{2}{n}\max\{p+1, q+1\}$. In the case $\sigma_1 \neq 0, \sigma_2 \neq 0$, Mochizuki and Huang [13] investigated the existence and nonexistence of global solutions, as well as the asymptotic behaviour of the global solution of problem (0.3), (0.4) on $R^n \times (0,+\infty)$. Levine [14] studied non-negative solutions of the initial–boundary value problem for system (0.3) with $\sigma_1 = \sigma_2 = 0$ in a domain $D \times (0,+\infty),$, where D is either a cone or the exterior of a bounded domain. Other results related to a system of type (0.3) can be found, for example, in [15]-[18], and the references therein.

In the present paper, we consider a system of three equations with singular potentials. Note that a Fujita-type result consisting of three equations was obtained in [19]. However, unlike that work, in the present study the equations also contain terms with singular potentials. The questions of existence and nonexistence of positive solutions for equations and systems of equations with singular potentials, both in the linear and nonlinear cases, have been investigated by various authors. For example, the celebrated result [20] by Baras and Goldstein established that the heat equation

$$\frac{\partial u}{\partial t}=\Delta u+\frac{C}{|x|^2}u, \text{ in } \Omega\times(0,T),\ u|_{\partial\Omega}=0$$

with singular inverse square potential in a smooth bounded domain $\Omega\subset R^n, n\geq 3$, such that $0\in\Omega$, in the supercritical range $C>\left(\frac{n-2}{2}\right)^2$ does not have a solution for any nontrivial L_1 initial data $u_0(x)\geq 0$.

The existence of global or local solutions of the semilinear equation $\frac{\partial u}{\partial t}-\Delta u+\frac{C}{|x|^2}u=u^q$, has also been studied by many authors in various domains. For example, in the work of Laptev and Hamidi [21], the existence of a global solution of this equation in the domain $R^n\times(0,\infty)$ was investigated for $C>-\left(\frac{n-2}{2}\right)^2$. Subsequently, in [22]-[26], the question of nonexistence of solutions for equations or inequalities of this type was investigated. In [27], a Fujita-type result was obtained in the domain $Q'_R=B'_R\times(0;+\infty)$ for a system of equations consisting of two such equations. In the present work, using the test function method, we obtain a sufficient condition for the nonexistence of nonnegative global solutions for systems of three equations.

2. The main result and its proof

Let's consider the following functions:

$$\varphi_0(s)\in C_0^\infty(R),\ \varphi_0(s)=\begin{cases}1, & s\leq 1\\ 0, & s\geq 2\end{cases},$$

$$\varphi(x)=\varphi_0^\kappa\left(\frac{|x|^2}{\rho^2}\right),\ T(t)=T\left(\frac{t}{\rho^2}\right)=\varphi_0^\gamma\left(\frac{t}{\rho^2}\right),$$

and

$$\xi_i(x)=|x|^{\lambda_i^+}-|x|^{\lambda_i^-},\ \lambda_i^\pm=-\frac{n-2}{2}\pm\sqrt{D_i},\ D_i=\left(\frac{n-2}{2}\right)^2-C_i,\ i=1,2,3.$$

It is easy to see that the functions $\xi_i(x)$ are radial solutions of the equations

$$\Delta v+\frac{C_i}{|x|^2}v=0 \quad \text{in} \quad B_1' \tag{1.1}$$

and $\xi_i(x)\big|_{|x|=1}=0$.

Let us introduce the following notations:

$$\eta_i=\sigma_i+2+q_i(\sigma_{i+1}+2)+q_iq_{i+1}(\sigma_{i-1}+2),$$

$$\theta_i=\frac{\eta_i}{q_1q_2q_3-1}-\lambda_i^+-n,$$

где $q_1=q_4, \sigma_1=\sigma_4, \sigma_0=\sigma_3$.

Before proceeding to the main result, let us agree that from now on we will denote all constants by C, despite the fact that in each specific case the constant may be different.

Our main result is the following theorem:

Теорема. Let $n\geq 3,\ q_i>1,\ C_i<\left(\frac{n-2}{2}\right)^2,\ \sigma_i\in R,\ u_{0i}(x)\geq 0,$ $u_{0i}\in L_{\infty,loc}\left(\overline{B}_R'\right),\ i=1,2,3.$ If $\max(\theta_1,\theta_2,\theta_3)\geq 0,$ then problem (0.1),(0.2) has no nonnegative solutions in Q_R'.

Proof. For simplicity of notation, we will take $R=1$. In the definition of the solution, we take as the test function $\eta(x,t)=\xi_i(x)\varphi(x)T(t)$. Then from the i-th equation of system (0.1) we obtain the following:

$$\iint\limits_{Q_1'}|x|^{\sigma_i}|u_{i+1}|^{q_i}\xi_i(x)\varphi(x)T(t)dxdt=-\iint\limits_{Q_1'}u_i\frac{\partial T}{\partial t}\xi_i\varphi dxdt-$$

$$-\iint\limits_{Q_1'}u_iT\left[\Delta(\xi_i\varphi)+\frac{C_i}{|x|^2}\xi_i\varphi\right]dxdt-\int\limits_{B_1'}u_{0i}(x)\xi_i(x)\varphi(x)dx+$$

$$+\int\limits_0^{+\infty}\int\limits_{|x|=1}u_iT\frac{\partial(\xi_i\varphi)}{\partial\nu}dsdt=-\iint\limits_{Q_1'}u_i\frac{\partial T}{\partial t}\xi_i\varphi dxdt-$$

$$-\iint\limits_{Q_1'}u_iT\varphi\left[\Delta\xi_i+\frac{C_i}{|x|^2}\xi_i\right]dxdt-\iint\limits_{Q_1'}u_iT\left[2(\nabla\xi_i,\nabla\varphi)+\xi_i\Delta\varphi\right]dxdt-$$

$$-\int_{B_1'} u_{0i}(x)\xi_i(x)\varphi(x)dx+\int_0^{+\infty}\int_{|x|=1} u_i\frac{\partial\xi_i}{\partial\nu}Tdsdt\le-\iint_{Q_1'} u_i\frac{\partial T}{\partial t}\xi_i\varphi dxdt-$$

$$-\iint_{Q_1'} u_iT\left[2(\nabla\xi_i,\nabla\varphi)+\xi_i\Delta\varphi\right]dxdt. \qquad (1.2)$$

Here we took into account that $\xi_i(x)$ is the solution of equations (1.1), $\int_{B_1'} u_{0i}(x)\xi_i(x)\varphi(x)dx\ge 0$ и $\int_0^{+\infty}\int_{|x|=1} u_i\frac{\partial\xi_i}{\partial\nu}Tdsdt=-\int_0^{+\infty}\int_{|x|=1} u_iT\frac{\partial\xi_i}{\partial r}dsdt\ge 0$, since $\frac{\partial\xi_i}{\partial r}\ge 0$ when $|x|=1$ and $u_i\ge 0$.

Using Hölder's inequality, from (1.2) we get that

$$\iint_{Q_1'} |x|^{\sigma_i}|u_{i+1}|^{q_i}\xi_i(x)\varphi(x)T(t)dxdt\le-\iint_{Q_1'} u_i\frac{\partial T}{\partial t}\xi_i\varphi dxdt+$$

$$+\iint_{Q_1'} u_iT\left[2(\nabla\xi_i,\nabla\varphi)+\xi_i\Delta\varphi\right]dxdt\le\left(\int_{\rho^2}^{2\rho^2}\int_{B_{1,\sqrt{2}\rho}}|x|^{\sigma_{i-1}}|u_i|^{q_{i-1}}\xi_{i-1}\varphi Tdxdt\right)^{\frac{1}{q_{i-1}}}\times$$

$$\times\left(\int_{\rho^2}^{2\rho^2}\int_{B_{1,\sqrt{2}\rho}}\frac{\left|\frac{\partial T}{\partial t}\right|^{q'_{i-1}}\xi_i^{q'_{i-1}}}{T^{(q'_{i-1}-1)}|x|^{\sigma_{i-1}(q'_{i-1}-1)}\varphi^{q'_{i-1}-1}\xi_{i-1}^{q'_{i-1}-1}}dxdt\right)^{\frac{1}{q'_{i-1}}}+$$

$$+\left(\int_0^{2\rho^2}\int_{B_{\rho,\sqrt{2}\rho}}|x|^{\sigma_{i-1}}|u_i|^{q_{i-1}}\xi_{i-1}\varphi Tdxdt\right)^{\frac{1}{q_{i-1}}}\times$$

$$\times\left(\int_0^{2\rho^2}\int_{B_{\rho,\sqrt{2}\rho}}\frac{\left|2(\nabla\xi_i,\nabla\varphi)+\xi_i\Delta\varphi\right|^{q'_{i-1}}T}{|x|^{\sigma_{i-1}(q'_{i-1}-1)}\varphi^{q'_{i-1}-1}\xi_{i-1}^{q'_{i-1}-1}}dxdt\right)^{\frac{1}{q'_{i-1}}}, \qquad (1.3)$$

where $q_0=q_3, q_0'=q_3', \sigma_0=\sigma_3, \xi_0=\xi_3$.

We denote the integral on the left-hand side of inequality (1.3) by A_i, the second integral of the first term on the right-hand side by I_{i-1}, and the second integral of the second term on the right-hand side by J_{i-1}. Then (1.3) can be written in the following form

$$A_i \le \left(\int_{\rho^2}^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_{i-1}} |u_i|^{q_{i-1}} \xi_{i-1} \varphi T dxdt \right)^{\frac{1}{q_{i-1}}} I_{i-1}^{\frac{1}{q'_{i-1}}} +$$

$$+ \left(\int_{0}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_{i-1}} |u_i|^{q_{i-1}} \xi_{i-1} \varphi T dxdt \right)^{\frac{1}{q_{i-1}}} J_{i-1}^{\frac{1}{q'_{i-1}}} . \qquad (1.4)$$

Increasing the boundaries of integration, we have that

$$A_i \le A_{i-1}^{\frac{1}{q_{i-1}}} I_{i-1}^{\frac{1}{q'_{i-1}}} + A_{i-1}^{\frac{1}{q_{i-1}}} J_{i-1}^{\frac{1}{q'_{i-1}}} = A_{i-1}^{\frac{1}{q_{i-1}}} \left(I_{i-1}^{\frac{1}{q'_{i-1}}} + J_{i-1}^{\frac{1}{q'_{i-1}}} \right). \qquad (1.5)$$

Now let's estimate the integrals I_{i-1}, J_{i-1}. To do this, we introduce the following substitutions:

$$t = \rho^2 s,\ r = \rho s,\ \xi_i(r) = \xi_i(\rho s) = \tilde{\xi}_i(s),\ \varphi(r) = \varphi(\rho s) = \varphi_0^{\kappa}(s^2) = \tilde{\varphi}(s),$$

$$T(t) = T(\rho^2 s) = \varphi_0^{\gamma}(s) = \tilde{T}(s). \qquad (1.6)$$

Since ξ_i, φ are radial functions, then

$$\frac{\partial \xi_i}{\partial x_j} = \frac{\partial \xi_i}{\partial r} \frac{x_j}{r}, \quad \frac{\partial \varphi}{\partial x_j} = \frac{\partial \varphi_i}{\partial r} \frac{x_j}{r},$$

$$\left(\nabla \xi_i, \nabla \varphi \right) = \sum_{j=1}^{n} \frac{\partial \xi_i}{\partial r} \frac{\partial \varphi}{\partial r} \left(\frac{x_j}{r}, \frac{x_j}{r} \right) = \frac{\partial \xi_i}{\partial r} \frac{\partial \varphi}{\partial r},\ \Delta \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} .$$

First, let's estimate J_{i-1}. Using substitution (1.6), we obtain that

$$J_{i-1} = \int_{0}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} \frac{\left| 2(\nabla \xi_i, \nabla \varphi) + \xi_i \Delta \varphi \right|^{q'_{i-1}} T}{|x|^{\sigma_{i-1}(q'_{i-1}-1)} \varphi^{q'_{i-1}-1} \xi_{i-1}^{q'_{i-1}-1}} dxdt \le$$

$$\le C \int_{0}^{2\rho^2} \int_{\rho}^{\sqrt{2}\rho} \frac{\left| 2 \frac{\partial \xi_i}{\partial r} \frac{\partial \varphi}{\partial r} + \xi_i \left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} \right) \right|^{q'_{i-1}} r^{n-1}}{r^{\sigma_i (q'_{i-1}-1)} \varphi^{q'_{i-1}-1} \xi_{i-1}^{q'_{i-1}-1}} drdt \le$$

$$\le C\rho^2 \int_1^{\sqrt{2}} \frac{\rho^{(\lambda_i^+ -2)q'_{i-1}} \left| 2\frac{\partial \tilde{\xi}_i}{\partial s}\frac{\partial \tilde{\varphi}}{\partial s} + \overleftarrow{\xi}_i \left(\frac{\partial^2 \tilde{\varphi}}{\partial s^2} + \frac{n-1}{s}\frac{\partial \tilde{\varphi}}{\partial s} \right) \right|^{q'_{i-1}} s^{n-1}\rho^n}{\rho^{\sigma_{i-1}(q'_{i-1}-1)} + \lambda_{i-1}^+ (q'_{i-1}-1)\cdot s^{\sigma_{i-1}(q'_{i-1}-1)} \tilde{\varphi}^{q'_{i-1}-1}} ds \le$$

$$\le C\rho^{2+(\lambda_i^+ -2)q'_{i-1} - \sigma_{i-1}(q'_{i-1}-1) - \lambda_{i-1}^+(q'_{i-1}-1)+n} \tilde{J}_{i-1} \le$$

$$\le C\rho^{-(q'_{i-1}-1)(\sigma_{i-1}+2+\lambda_{i-1}^+ - \lambda_i^+)+\lambda_i^+ +n} \tilde{J}_{i-1}, \tag{1.7}$$

where

$$\tilde{J}_{i-1} = \int_1^{\sqrt{2}} \frac{\left| 2\frac{\partial \tilde{\xi}_i}{\partial s}\frac{\partial \tilde{\varphi}}{\partial s} + \tilde{\xi}_i \left(\frac{\partial^2 \tilde{\varphi}}{\partial s^2} + \frac{n-1}{s}\frac{\partial \tilde{\varphi}}{\partial s} \right) \right|^{q'_{i-1}} s^{n-1}}{s^{\sigma_{i-1}(q'_{i-1}-1)} \tilde{\varphi}^{q'_{i-1}-1}} ds .$$

It is easy to show that for large values κ the integral $\tilde{J}_{i-1}$ is bounded (see [1]). Then, from (1.7) it follows that

$$J_{i-1}^{\frac{1}{q'_{i-1}}} \le C\rho^{-\frac{1}{q_{i-1}}[(\sigma_{i-1}+2+\lambda_{i-1}^+ - \lambda_i^+)-(\lambda_i^+ +n)(q_{i-1}-1)]} = C\rho^{-\frac{\alpha_{i-1}}{q_{i-1}}}, \tag{1.8}$$

where

$$\alpha_{i-1} \equiv \sigma_{i-1} + 2 + \lambda_{i-1}^+ - \lambda_i^+ - (\lambda_i^+ + n)(q_{i-1}-1),\ \lambda_0^+ = \lambda_3^+, \alpha_0 = \alpha_3.$$

Now let us estimate I_{i-1}.

$$I_{i-1} = \int_0^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} \frac{\left|\frac{\partial T}{\partial t}\right|^{q'_{i-1}} \xi_i^{q'_{i-1}}}{T^{(q'_{i-1}-1)} |x|^{\sigma_{i-1}(q'_{i-1}-1)} \varphi^{q'_{i-1}-1} \xi_{i-1}^{q'_{i-1}-1}} dxdt =$$

$$= \int_{\rho^2}^{2\rho^2} \frac{\left|\frac{\partial T}{\partial t}\right|^{q'_{i-1}}}{T^{(q'_{i-1}-1)}} dt \int_{B_{1,\sqrt{2}\rho}} \frac{\xi_i^{q'_{i-1}}}{|x|^{\sigma_{i-1}(q'_{i-1}-1)} \varphi^{q'_{i-1}-1} \xi_{i-1}^{q'_{i-1}-1}} dx. \tag{1.9}$$

Using substitution (1.6), we estimate these integrals as follows:

$$\int_{\rho^2}^{2\rho^2} \frac{\left|\frac{\partial T}{\partial t}\right|^{q'_{i-1}}}{T^{(q'_{i-1}-1)}} dt \le C \int_1^2 \frac{\rho^{-2q'_{i-1}} \left|\frac{\partial \tilde{T}}{\partial \tau}\right|^{q'_{i-1}}}{\tilde{T}^{q'_{i-1}-1}} \rho^2 d\tau \le$$

$$\le C\rho^{-2(q'_{i-1}-1)}\int_1^2 \frac{\left|\frac{\partial\widetilde{T}}{\partial\tau}\right|^{q'_{i-1}}}{\widetilde{T}^{q'_{i-1}}}\,d\tau \le C\rho^{-2(q'_{i-1}-1)}\widetilde{I}_{i-1},$$

where the integral $\int_1^2 \frac{\left|\frac{\partial\widetilde{T}}{\partial\tau}\right|^{q'_{i-1}}}{\widetilde{T}^{q'_{i-1}-1}}\,dt$ is denoted by $\widetilde{I}_{i-1}$.

$$\int_{B_{1,\sqrt{2}\rho}} \frac{\xi_i^{q'_{i-1}}}{|x|^{\sigma_{i-1}(q'_{i-1}-1)}\varphi^{q'_{i-1}-1}\xi_{i-1}^{q'_{i-1}-1}}\,dx \le$$

$$\le C\int_1^{\sqrt{2}\rho} \frac{r^{\lambda_i^+ q'_{i-1}}\left(1-r^{2\sqrt{D_i}}\right)^{q'_{i-1}} r^{n-1}}{r^{\sigma_{i-1}(q'_{i-1}-1)}\varphi^{q'_{i-1}-1} r^{\lambda_{i-1}^+(q'_{i-1}-1)}\left(1-r^{2\sqrt{D_{i-1}}}\right)^{q'_{i-1}}}\,dr \le$$

$$\le C\begin{cases}\rho^{-\beta_{i-1}(q'_{i-1}-1)}, & for\ \beta_{i-1}<0\\ \ln 2\rho, & for\ \beta_{i-1}=0\\ 1, & for\ \beta_{i-1}>0,\end{cases}$$

where $\beta_{i-1}=\sigma_{i-1}+\lambda_{i-1}^+-\lambda_i^+-(\lambda_i^+ +n)\frac{1}{(q'_{i-1}-1)}=$

$$=\sigma_{i-1}+\lambda_{i-1}^+-\lambda_i^+-(\lambda_i^+ +n)(q_{i-1}-1)=\alpha_{i-1}-2.$$

As a result, from the last two estimates it follows that

$$I_{i-1}\le C\widetilde{I}_{i-1}\begin{cases}\rho^{-(\alpha_{i-1})(q'_{i-1}-1)}, & for\ \beta_{i-1}<0\\ \rho^{-2(q'_{i-1}-1)}\ln 2\rho, & for\ \beta_{i-1}<0\\ \rho^{-2(q'_{i-1}-1)}, & for\ \beta_{i-1}<0\end{cases}$$

It is also easy to show that for large γ, the integral $\widetilde{I}_{i-1}$ is bounded (see [1]). Therefore,

$$I_{i-1}^{\frac{1}{q'_{i-1}}} \le C\begin{cases} \rho^{-\frac{\alpha_{i-1}}{q_{i-1}}}, & for\ \beta_{i-1} < 0 \\ \rho^{-\frac{2}{q_{i-1}}}(\ln 2\rho)^{\frac{1}{q'_{i-1}}}, & for\ \beta_{i-1} = 0 \\ \rho^{-\frac{2}{q_{i-1}}}, & for\ \beta_{i-1} > 0, \end{cases} \tag{1.10}$$

Let us denote $B_{i-1} \equiv \rho^{-\frac{\alpha_{i-1}}{q_{i-1}}}, B_0 = B_3$ and consider the case when all $\beta_i < 0$. Then we can write (1.8) and (1.10) in the following form:

$$I_{i-1}^{\frac{1}{q'_{i-1}}} \le CB_{i-1},\ J_{i-1}^{\frac{1}{q'_{i-1}}} \le CB_{i-1}.$$

Taking these estimates into account in (1.4) and (1.5), we get:

$$A_i \le C\left[\left(\int_{\rho^2}^{2\rho^2}\int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_{i-1}}|u_i|^{q_{i-1}}\xi_{i-1}\varphi T dxdt\right)^{\frac{1}{q_{i-1}}} + \right.$$

$$\left. + \left(\int_{0}^{2\rho^2}\int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_{i-1}}|u_i|^{q_{i-1}}\xi_{i-1}\varphi T dxdt\right)^{\frac{1}{q_{i-1}}}\right] B_{i-1}, \tag{1.11}$$

$$A_i \le CA_{i-1}^{\frac{1}{q_{i-1}}} B_{i-1}. \tag{1.12}$$

From now on, we agree that, when necessary, the index $i-2$ will be understood as $i+1$, $i+2$ as $i-1$, and $i\pm 3$ as i.

Using the recurrence formula (1.12), we obtain the following:

$$A_i \le CA_{i-1}^{\frac{1}{q_{i-1}}} B_{i-1} \le C\left(A_{i+1}^{\frac{1}{q_{i+1}}} B_{i+1}\right)^{\frac{1}{q_{i-1}}} B_{i-1} \le CA_{i+1}^{\frac{1}{q_{i-1}q_{i+1}}} B_{i+1}^{\frac{1}{q_{i-1}}} B_{i-1}$$

$$\le C\left(A_i^{\frac{1}{q_i}} B_i\right)^{\frac{1}{q_{i-1}q_{i+1}}} B_{i+1}^{\frac{1}{q_{i-1}}} B_{i-1} \le A_i^{\frac{1}{q_{i-1}q_iq_{i+1}}} B_i^{\frac{1}{q_{i-1}q_{i+1}}} B_{i+1}^{\frac{1}{q_{i-1}}} B_{i-1}, \tag{1.13}$$

If we divide both sides of (1.13) by $A_i^{\frac{1}{q_{i-1}q_iq_{i+1}}}$ and raise them to the power

$q_{i-1}q_iq_{i+1}$, then in the end we obtain that

$$A_i^{q_1q_2q_3-1} \le CB_i^{q_i} B_{i+1}^{q_iq_{i+1}} B_{i-1}^{q_{i-1}q_iq_{i+1}} \le CB_i^{q_i} B_{i+1}^{q_iq_{i+1}} B_{i-1}^{q_1q_2q_3}.$$

From this, taking into account the value of B_i, we get the following:

$$A_i^{q_1q_2q_3-1} \le C\rho^{-\alpha_i} \rho^{-\alpha_{i+1}q_i} \rho^{-\alpha_{i-1}q_iq_{i+1}} \le C\rho^{-[\alpha_i+\alpha_{i+1}q_i+\alpha_{i-1}q_iq_{i+1}]}. \qquad (1.14)$$

Let us compute the exponent of ρ in (1.14).

$$\alpha_i+\alpha_{i+1}q_i+\alpha_{i-1}q_iq_{i+1} = \sigma_i+2+\lambda_i^+-\lambda_{i+1}^+-(\lambda_{i+1}^++n)(q_i-1)+$$
$$+q_i(\sigma_{i+1}+2+\lambda_{i+1}^+-\lambda_{i-1}^+-(\lambda_{i-1}^++n)(q_{i+1}-1))+$$
$$+q_iq_{i+1}(\sigma_{i-1}+2+\lambda_{i-1}^+-\lambda_i^+-(\lambda_i^++n)(q_{i-1}-1))=$$
$$=\sigma_i+2+\lambda_i^+-\lambda_{i+1}^++\lambda_{i+1}^+-q_i\lambda_{i+1}^+-n(q_i-1)+q_i(\sigma_{i+1}+2)+q_i\lambda_{i+1}^+-q_i\lambda_{i-1}^+-$$
$$-q_iq_{i+1}\lambda_{i-1}^++q_i\lambda_{i-1}^+-nq_i(q_{i+1}-1)+q_iq_{i+1}(\sigma_{i-1}+2)+\lambda_{i-1}^+q_iq_{i+1}-$$
$$-\lambda_i^+q_iq_{i+1}+\lambda_i^+q_iq_{i+1}-\lambda_i^+q_{i-1}q_iq_{i+1}-nq_iq_{i+1}(q_{i-1}-1)=$$
$$=\sigma_i+2+q_i(\sigma_{i+1}+2)+q_iq_{i+1}(\sigma_{i-1}+2)-$$
$$-n(q_i-1)-nq_i(q_{i+1}-1)-nq_iq_{i+1}(q_{i-1}-1)-$$
$$-\lambda_i^+(q_{i-1}q_iq_{i+1}-1)=\sigma_i+2+q_i(\sigma_{i+1}+2)+q_iq_{i+1}(\sigma_{i-1}+2)-$$
$$-n(q_1q_2q_3-1)-\lambda_i^+(q_1q_2q_3-1).$$

Taking this into account, from (1.14) we have that

$$A_i \le C\rho^{-\left(\frac{\eta_i}{q_1\ldots q_k-1}-\lambda_i^+-n\right)} = C\rho^{-\theta_i}, \qquad (1.15)$$

where

$$\eta_i=\sigma_i+2+q_i(\sigma_{i+1}+2)+q_iq_{i+1}(\sigma_{i-1}+2),\ \theta_i=\frac{\eta_i}{q_1q_2q_3-1}-\lambda_i^+-n.$$

Let now $\max\{\theta_1,\theta_2,\theta_3\}\ge 0$. First, consider the case when $\max\{\theta_1,\theta_2,\theta_3\}>0$. For clarity, let's take $\theta_1>0$. Then, taking $i=1$ in (1.15), we obtain that

$$A_1 \le C\rho^{-\theta_1}. \qquad (1.16)$$

Letting ρ tend to $+\infty$, we get that

$$A_1 \le 0.$$

This means

$$\iint_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 dxdt \le 0.$$

Hence $u_2 \equiv 0$. Then from the second equation it follows that $u_3 \equiv 0$, and from the next equation $u_1 \equiv 0$.

Now let $\max\{\theta_1, \theta_1, \theta_3\} = 0$. For clarity, let's take $\theta_1 = \eta_1 - (\lambda_1^+ + n)(q_1 q_2 q_3 - 1) = 0$. Then from (1.16) we have that $A_1 \le C$. Since $A_1 \le C$, it follows from the property of the integral that when $\rho \to +\infty$

$$\int_{\rho^2}^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1(x)\varphi(x)T(t)dxdt \to 0, \tag{1.17}$$

$$\int_{1}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1(x)\varphi(x)T(t)dxdt \to 0. \tag{1.18}$$

Using (1.13) from (1.11), we obtain that

$$A_i \le C A_{i+1}^{\frac{1}{q_{i-1}q_{i+1}}} B_{i+1}^{\frac{1}{q_{i-1}}} B_{i-1} \le C \left[\left(\int_{\rho^2}^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_i} |u_{i+1}|^{q_i} \xi_i \varphi T dxdt \right)^{\frac{1}{q_1 q_2 q_3}} + \right.$$

$$\left. + \left(\int_{0}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_i} |u_{i+1}|^{q_i} \xi_i \varphi T dxdt \right)^{\frac{1}{q_1 q_2 q_3}} \right] B_i^{\frac{1}{q_{i-1}q_{i+1}}} B_{i+1}^{\frac{1}{q_{i-1}}} B_{i-1}.$$

Raising each side of this inequality to the power $q_1 q_2 q_3$, as in (1.15), we have that

$$A_i^{q_1 q_2 q_3} \le C \left[\int_{\rho^2}^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_i} |u_{i+1}|^{q_i} \xi_i \varphi T dxdt + \int_{0}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_i} |u_{i+1}|^{q_i} \xi_i \varphi T dxdt \right] \times$$

$$\times \rho^{-\eta_i + (\lambda_i^+ + n)(q_1 q_2 q_3 - 1)}. \tag{1.19}$$

If we take $i = 1$ in (1.19) and take into account that $-\eta_1 + (\lambda_1^+ + n)(q_1 q_2 q_3 - 1) = 0$, then we obtain the following

$$A_1^{q_1 q_2 q_3} \le C \left[\int_{\rho^2}^{2\rho^2} \int_{B_{1,\sqrt{2}\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi T dxdt + \int_{0}^{2\rho^2} \int_{B_{\rho,\sqrt{2}\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi T dxdt \right].$$

Hence, passing to the limit as $\rho \to +\infty$ and using the properties of integrals

(1.17), (1.18), we have that

$$\iint\limits_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1(x)dxdt \le 0.$$

Therefore, $u_2 \equiv 0$. Then, as in the previous case, we obtain that all $u_i \equiv 0, \quad i = 1,2,3$.

Finally, it should be noted that we considered the case $\beta_{i-1} < 0$; however, since the power of ρ is negative in the case $\beta_{i-1} > 0$, it can be similarly and easily shown that the $u_i \equiv 0,\ i = 1,2,3$ values are identically zero. This completes the proof of the theorem.

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