Baku State University Journal of Mathematics & Computer Sciences ***2025****, v. 2 (4), p. 1-19*

journal homepage: http://bsuj.bsu.edu.az/en

GLOBAL BIFURCATION OF SOLUTIONS FROM ZERO AND INFINITY OF SOME NONLINEAR EIGENVALUE PROBLEMS WITH SPECTRAL PARAMETER IN TWO OF BOUNDARY CONDITIONS

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Received 18 septembre 2025; accepted 10 october 2025

Abstract

We consider global bifurcation of nonlinear eigenvalue problems for ordinary differential equations fourth order with a spectral parameter in two of boundary conditions. The existence of two pairs of two families of global continua of solutions, emanating from intervals of the line of trivial solutions and the line $R \times \{\infty\},$ which are contained in classes of functions that have oscillatory properties of eigenfunctions of a linear problem in the neighborhoods of these intervals, is established.

Keywords: nonlinear eigenvalue problem, global bifurcation, bifurcation from zero, bifurcation from infinity, global continua, oscillation properties of eigenfunction.

1. Introduction

In this paper, we continue to study the bifurcation of nontrivial solutions of the following nonlinear eigenvalue problem

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$$\ell(y) \equiv y^{(4)} - (q(x)y')' = \lambda y + h(x, y, y', y'', y''', \lambda), \; x \in (0, 1), \tag{1}$$

$$y''(0) = y''(1) = 0, \tag{2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{3}$$

$$Ty(1) - c\lambda y(0) = 0, \tag{4}$$

where $\lambda \in R$ is an eigenvalue parameter, $Ty \equiv y''' - qy'$, $q(x)$ is a positive and absolutely continuous function on $[0, 1]$, a, c are real constants such that $a > 0$ and $c < 0$. We suppose that the nonlinear term has the form $h = f + g$, where f and g are real-valued continuous functions on $[0, 1] \times R^5$ satisfying the following conditions: there exists positive constant M such that

$$\frac{|f(x, y, s, \upsilon, w, \lambda)|}{|y|} \le M, \; x \in [0, 1], (y, s, \upsilon, w) \in R^4, y \ne 0, \; \lambda \in R; \tag{5}$$

for every bounded interval $\Lambda \subset R$,

$$g(x, y, s, \upsilon, w, \lambda) = o(|y| + |s| + |\upsilon| + |w|) \text{ as } |y| + |s| + |\upsilon| + |w| \to 0, \tag{6}$$

or

$$g(x, y, s, \upsilon, w, \lambda) = o(|y| + |s| + |\upsilon| + |w|) \text{ as } |y| + |s| + |\upsilon| + |w| \to +\infty, \tag{7}$$

uniformly for $(x, \lambda) \in [0,1] \times R$.

We recall that the global bifurcation from zero and infinity of nontrivial solutions of nonlinear eigenvalue problems for ordinary differential equations of the second and fourth orders was studied in detail in [1-4, 6-8, 11, 12, 17-20]. Note that in [1, 3, 4, 6, 7, 12] one of the boundary conditions of considered problems contains an eigenvalue parameter. Therefore, in these articles, using angular functions, classes of functions with oscillatory properties of eigenfunctions of the corresponding linear problems are constructed.

In this paper, the construction of oscillatory classes is refined, the results of paper [16] are strengthened, and, in addition, a global bifurcation from infinity is studied, as well as a global bifurcation from zero and infinity simultaneously.

It should be noted that the bifurcation of eigenvalue problems for ordinary differential equations describes the form of instability of various processes in physics and mechanics. In connection with the physical meaning of the problem under consideration, (1)-(4), one can find, for example, in [13].

2. Preliminary results and operator interpretation of problem (1)-(4)

We consider the following linear spectral problem

$$\begin{cases} \ell(y) = \lambda y \;\; x \in (0,1), \\ y''(0) = y''(1) = 0, \\ Ty(0) - a\lambda y(0) = 0, \\ Ty(1) - c\lambda y(0) = 0. \end{cases} \tag{8}$$

Problem (8) was studied in detail in [9], where, in particular, it was shown that the eigenvalues of this problem are real and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that

$$0 = \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots .$$

Moreover, for each $k \in \mathrm{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k has $k-1$ simple zeros in the interval $(0, 1)$.

The linear eigenvalue problem (8) is reduced to the eigenvalue problem in the Hilbert space $H = L_2(0,1) \oplus C^2$ with the scalar product

$$(\hat{y}, \hat{\upsilon})_H = (\{y, m, n\}, \{\upsilon, s, t\})_H = \int_0^1 y(x)\overline{\upsilon(x)}dx + a^{-1}m\bar{s} - c^{-1}n\bar{t},$$

for the operator L defined by

$$L\hat{y} = L\{y, m, n\} = \{\ell(y), Ty(0), Ty(1)\}$$

on the domain

$$D(L) = \{\hat{y} = \{y, m, n\} \in H : y,\, qy', y'', Ty \in AC[0,1],\; \ell(y) \in L_2(0,1),$$
$$y''(0) = y''(1) = 0, m = ay(0), n = ay(1)\}$$

which is dense in H [9]. It is obvious that the operator L is defined correctly and problem (8) is equivalent to the linear eigenvalue problem

$$L\hat{y} = \lambda\hat{y},\; \hat{y} \in D(L), \tag{9}$$

i.e. the eigenvalues $\lambda_k, k \in \mathrm{N},$ of problems (8) and (9) coincide, and there is a one-to-one correspondence between their eigenfunctions and eigenvectors:

$$y_k \leftrightarrow \hat{y}_k = \{y_k, m_k, n_k\},\; m_k = ay_k(0),\; n_k = cy_k(1).$$

For each $y \in C^3[0,1]$ and each $\{m, n\} \in R^2$ let

$$|\, y\,|_3 = |\, y\,|_0 + |\, y'\,|_0 + |\, y''\,|_0 + |\, y'''\,|_0,$$

$$\| y \|_3 = | y |_3 + | m | + | n |, \tag{10}$$

where $| y |_0 = \max\limits_{x \in [0,1]} | y(x) |$.

Let $E = C^3[0,1] \bigcap (b.c.)$ be the Banach space with the norm $| y |_3$, where $(b.c.)$ is a set of functions which satisfy boundary conditions (2).

Since by (2) the statements of Lemma 1.2 of [10] hold for problem (8), to study the bifurcation of problem (1)-(4), as in [12], we introduce a subset S of the space E as follows:

$S = \{ u \in E : u^{(i)}(x) \neq 0, \, Tu(x) \neq 0, \, x \in [0,1], \, i = 0,1,2 \} \bigcup \{ u \in E : \exists i_0 \in \{0,1,2\},$

$x_0 \in (0,1)$ such that $u^{(i_0)}(x_0) = 0 \vee Tu(x_0) = 0,$ and if $u'(x_0) Tu(x_0) = 0,$ then

$u(x) u''(x) < 0$ in a neighbourhood of x_0, and if $u(x_0) u''(x_0) = 0,$ then

$u'(x) Tu(x) < 0$ in a neighbourhood of $x_0 \} \bigcup \{ y \in E : | y(j) | + | y'(j) | +$

$$+ | y''(j) | + | y'''(j) | > 0, \, j = 0,1 \}.$$

Suppose that $\hat{E}$ is a Banach space $E \oplus R^2$ with norm given by formula (10) and

$$\hat{S} = \{ \hat{y} \in \hat{E} : y \in S \}.$$

The elements of the set $\hat{S}$ are vectors $\hat{y} = \{ y, m, n \} \in \hat{E}$ such that the functions $y(x)$, which are the first coordinates of these vectors, have a finite number of zeros in the interval [0,1], so that the zeros located in the interval $(0,1)$ are simple, and the multiplicity of zeros at the ends of the interval cannot be more than four. If $\hat{y} = \{ y, m, n \} \in D(L),$ then by $qy' \in AC[0,1]$ and $Ty \in AC[0,1]$ it follows from the relation $y''' = Ty + qy'$ that $y''' \in C[0,1]$ or $y \in C^3[0,1]$. As a result, we obtain the relation $\hat{y} = \{ y, m, n \} \in \hat{E},$ which shows that $D(L) \subseteq \hat{E}$.

Now we define nonlinear operators $F : R \times \hat{E} \to C[0,1] \oplus R^2$ and $G : R \times \hat{E} \to C[0,1] \oplus R^2$ as follows:

$$F(\lambda, \hat{y}) = F(\lambda, \{ y, m, n \}) = \{ f(x, y, y', y'', y''', \lambda), 0, 0 \},$$

$$G(\lambda, \hat{y}) = G(\lambda, \{ y, m, n \}) = \{ g(x, y, y', y'', y''', \lambda), 0, 0 \},$$

where $C[0,1] \oplus R^2$ is a Banach space with the norm $\| y \|_0 = | y |_0 + | m | + | n |$. Then we can write the nonlinear eigenvalue problem (1)-(4) as the following equivalent operator equation:

$$L\hat{y} = \lambda\hat{y} + F(\lambda,\hat{y}) + G(\lambda,\hat{y}). \tag{11}$$

Note that between the solutions of nonlinear eigenvalue problems (1)-(4) and (11) there is a following one-to-one correspondence:

$$(\lambda, y) \leftrightarrow (\lambda,\hat{y}) = (\lambda,\{y,m,n\}),\ m = ay(0),\ n = cy(1). \tag{12}$$

3. Prüfer transformation for the linear eigenvalue problem (1)-(4) with $h \equiv 0$

As in [2- 4], we will use the following Prüfer-type transformation to study the bifurcation of solutions to problem (1)-(4):

$$\begin{cases} y(x) = \rho(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = \rho(x)\cos\psi(x)\sin\phi(x), \\ y''(x) = \rho(x)\cos\psi(x)\cos\phi(x), \\ Ty(x) = \rho(x)\sin\psi(x)\sin\theta(x). \end{cases} \tag{13}$$

Direct calculations show that for the Jacobian $J(y)$ of transformation (13) the following expression hold:

$$J(y)(x) = \rho^3(x)\sin\psi(x)\cos\psi(x),\ x\in[0,1]. \tag{14}$$

According to relations (1.6) in [10], the angular functions θ and φ satisfy the following system of equations:

$$\theta' = -w\sin\varphi\sin\theta + \lambda\cos^2\theta, \tag{15}$$

$$\varphi' = \cos^2\varphi - q\sin^2\varphi - \frac{1}{w}\sin\theta\sin\varphi, \tag{16}$$

where $w = \cot\psi$.

If the function $y(x,\lambda),\ x\in[0,1],\ \lambda\in R,$ is a solution to equation in (8), then by the uniqueness property of the solution to the Cauchy problem, we obtain that

$$\rho(x,\lambda) \neq 0,\ x\in[0,1],\ \lambda\in R. \tag{17}$$

of the transformation (13) does not vanish in $(0,\ 1)$.

Remark 1. If the function $y(x,\lambda),\ x\in[0,1],\ \lambda\in R,$ is a solution of equation in (6) satisfying the boundary conditions (2) for $\lambda > 0,$ then by [10, Lemma 1.2], the following inequality holds:

$$\cos\psi(x,\lambda)\sin\psi(x,\lambda)\neq 0,\ x\in(0,1). \tag{18}$$

This, together with (17), shows that for $\lambda>0,$ the relation

$$J(y)(x,\lambda)\neq 0,\ x\in(0,1). \tag{19}$$

holds. Moreover, by (18) for the function $w(x,\lambda)=\cot\psi(x,\lambda),\ x\in(0,1),$ we have one of the following relations is true:

$$w(x,\lambda)\in(0,+\infty)\ \text{ or }\ w(x,\lambda)\in(-\infty,0).$$

Therefore, without loss of generality, we can assume that one of the relations

$$\psi(x,\lambda)\in\left(0,\pi/2\right)\ \text{ or }\ \psi(x,\lambda)\in\left(\pi/2,\pi\right) \tag{20}$$

holds.

Lemma 1. *Assume that the function $y(x,\lambda)$ is a solution of equation in (8) satisfying the boundary conditions (2) and (3) for $\lambda>0.$ Then, for each $\lambda>0,$ the initial conditions are satisfied:*

$$\theta(0,\lambda)=\tan^{-1}a\lambda\in\left(0,\pi/2\right), \tag{21}$$

$$\varphi(0,\lambda)=\frac{\pi}{2}. \tag{22}$$

Moreover, for $w(0,\lambda)$ we have the following relation:

$$w(0,\lambda)=\frac{y(0,\lambda)}{y'(0,\lambda)}\frac{1}{\sqrt{1+a^2\lambda^2}}. \tag{23}$$

Remark 2. If $\theta=k\pi,k\in\mathrm{Z},$ then by (15) we have $\theta'=\lambda,$ and consequently, if $\lambda>0,$ then the graph of the function $\theta(x,\lambda),\ x\in(0,1),$ intersects the lines $\theta=k\pi,k\in\mathrm{Z},$ in the positive slope, that is, this function takes increasing values of $k\pi,\ k\in\mathrm{Z}.$ Moreover, if $\lambda>0,$ then by (15) and [10, Theorem 5.2] the graph of $\theta(x,\lambda),\ x\in(0,1),$ intersects the lines $\theta=-\pi/2+k\pi,k\in\mathrm{Z},$ in the positive slope.

Remark 3. If $\varphi=k\pi,k\in\mathrm{Z},$ then by (16) we have $\varphi'=1,$ and consequently, if $\lambda>0,$ then the graph of the function $\varphi(x,\lambda),\ x\in(0,1),$ intersects the lines $\varphi=k\pi,k\in\mathrm{Z},$ in the positive slope [10, theorem 6.1].

Using Remarks 2 and 3 by Lemma 1 we have the following result.

Lemma 2. *Let the function $y(x,\lambda)$ be a solution of equation in (8) satisfying the boundary conditions (2) and (4) for $\lambda>0.$ Then, for each $\lambda>0,$ the following relation* *holds:*

$$\theta(1,\lambda)=\tan^{-1}c\lambda\in\left(-\pi/2+\pi k,\pi k\right), \tag{24}$$

$$\varphi(0,\lambda)=-\pi/2+\pi k,\ k=1,\,2,\,...\,. \tag{25}$$

Theorem 1. *Let* $y(x,\lambda)$ *be a solution of equation in (8) satisfying the boundary conditions (2) and (3) for* $\lambda>0,$ *and* $\theta(x,\lambda)$ *is the* θ *-angle defined by the transformation (13). Then the function* $\theta(1,\lambda)$ *is a strictly increasing continuous function of the parameter* $\lambda>0.$

The proof of this theorem is similar to that of [10, theorem 4.2].

Remark 4. By [9, Theorem 4.5] $\lambda=0$ is the first eigenvalue of problem (8), and the corresponding eigenfunction is $y(x,\lambda)=y(x,0)\equiv const.$ Then, by transformation (13), $\sin\theta(x,0)\equiv 0,$ and consequently, for some $m\in\mathrm{Z}$ we have $\theta(x,0)\equiv\pi m.$ By (21) $\theta(0,0)=\arctan 0=0\in\left[0,\pi/2\right]$ which implies that $\theta(x,0)\equiv 0.$ In this case, we can assume that $\theta(y_1,0)=0.$

Remark 5. Since $c<0$ by Lemmas 1 and 2, Theorem 1 and Remark 4 for each $k=1,2,\,...\,,$ the equation

$$\theta(1,\lambda)=\arctan c\lambda+\pi k,\ \lambda>0,$$

has unique solution $\lambda=\lambda_{k+2}.$ Therefore, using Remark 4 for the eigenvalues of problem (8) we have the following relation:

$$\theta(1,\lambda_k)=\arctan c\lambda_k+\pi k,\ k\in N. \tag{26}$$

4. The classes $\hat{S}_k^{\nu}$ and $\hat{S}_k^{\nu},\ k\in\mathrm{N},\nu\in\{+,-\}$

To study the bifurcation of solutions of the nonlinear eigenvalue problem (1)-(4), as in [2], we construct the sets $\hat{S}_k^{\nu}$ and $\hat{S}_k^{\nu},\ k\in\mathrm{N},\nu\in\{+,-\}$, using angular functions presented in Section 2.

If $y\in S,$ then it follows from (13) that

$$J(y)(x)\neq 0,\ x\in(0,\,1).$$

For each $\lambda\in R$ and each $y\in S$ we define continuous on [0,1] functions as follows:

$$\rho^2(\lambda, y, x) = y^2(x,\lambda) + y'^2(x,\lambda) + y''^2(x,\lambda) + (Ty(x,\lambda))^2,$$

$$\theta(\lambda, y, x) = \arctan\frac{Ty(x,\lambda)}{y(x,\lambda)},\ \theta(\lambda, y, 0) = \arctan a\lambda,$$

$$\varphi(\lambda, y, x) = \arctan\frac{y'(x,\lambda)}{y''(x,\lambda)},\ \varphi(\lambda, y, 0) = \pi/2,$$

$$w(\lambda, y, x) = \cot\frac{y'(x,\lambda)}{y''(x,\lambda)} \in (\pi/2, \pi),\ w(\lambda, y, 0) = \frac{y'(0,\lambda)}{y(0,\lambda)}\frac{1}{\sqrt{1+a^2\lambda^2}}.$$

For each $k \in \mathrm{N}$, each $\nu \in \{+,-\}$ and each $\lambda \in R$, let $S^{\nu}_{k,\lambda}$ denote the set of functions $y \in S$ that satisfy the following conditions:

(i) $\theta(\lambda, y, 0) = \beta(\lambda)$;

(ii) $\varphi(\lambda, y, 0) = \frac{\pi}{2},\ \varphi(\lambda, y, 1) = \frac{\pi}{2} + \pi(k-1),$

(iii) $\theta(\lambda, y, 0) = \gamma(\lambda) + \pi(k-1)$;

(iv) for each fixed λ and each fixed y, as the variable x increases from 0 to 1, the function $\theta(\lambda, y, x)$ increases takes the values $m\pi/2,\ m \in \mathrm{Z}$; as the variable x decreases from 1 to 0, the function $\theta(\lambda, y, x)$ decreases takes the values $m\pi/2,\ m \in \mathrm{Z}$;

(v) for each fixed λ and each fixed y, as the variable x increases from 0 to 1, the function $\varphi(\lambda, y, x)$ increases takes the values $s\pi,\ s \in \mathrm{Z}$; as the variable x decreases from 1 to 0, the function $\varphi(\lambda, y, x)$ decreases takes the values $s\pi,\ s \in \mathrm{Z}$;

(vi) the function $\nu y(x)$ is positive in the right punctured neighborhood of the point $x = 0$.

It follows from Section 2 that for each $k \in \mathrm{N}$, each $\nu \in \{+, -\}$ and each $\lambda \geq 0$ the sets $S^{+}_{k,\lambda}$, $S^{-}_{k,\lambda}$ and $S_{k,\lambda} = S^{+}_{k,\lambda} \bigcup S^{-}_{k,\lambda}$ are nonempty. By definition, for each $k \in \mathrm{N}$, each $\nu \in \{+, -\}$ and each $\lambda \in R$ these sets are open subsets in E, and for each $(k', \nu') \neq (k, \nu)$ the relations hold:

$$S^{\nu'}_{k',\lambda} \bigcap S^{\nu}_{k,\lambda} = \varnothing \quad \text{and} \quad S_{k',\lambda} \bigcap S_{k,\lambda} = \varnothing.$$

Moreover, it follows from [1, Lemma 2.2] that if $\lambda \in R$ is fixed and $y \in \partial S^{\nu}_{k,\lambda}\ (\partial S_{k,\lambda})$, then there exists $\xi \in [0,1]$ such that $y(\xi) = y'(\xi) = y''(\xi) = \ = y'''(\xi) = 0.$

Now for each $k \in \mathrm{N}$ and each $\nu \in \{+,-\}$ we define subsets S_k^ν and S_k of the set $S \subset E$ as follows:

$$S_k^\nu = \bigcup_{\lambda \in R} S_{k,\lambda}^\nu \quad \text{and} \quad S_k = \bigcup_{\lambda \in R} S_{k,\lambda}.$$

Then for each for each $k \in \mathrm{N}$ and each $\nu \in \{+,-\}$ these sets are open subsets in E, and for each $(k',\nu') \neq (k,\nu)$ the relations hold:

$$S_{k'}^{\nu'} \bigcap S_k^\nu = \varnothing \quad \text{and} \quad S_{k'} \bigcap S_k = \varnothing.$$

Moreover, if $y \in \partial S_k^\nu \, (\partial S_k),$ then there exists $\xi \in [0,1]$ such that $y(\xi) = y'(\xi) =$ $y''(\xi) = y'''(\xi) = 0.$

Finally, for each $k \in \mathrm{N}$ and each $\nu \in \{+,-\},$ we introduce subsets $\hat{S}_k^\nu$ and $\hat{S}_k$ of the set $\hat{S} \subset \hat{E}$ as follows:

$$\hat{S}_k^\nu = \{\hat{y} = \{y,m,n\} \in \hat{E} \colon y \in S_k^\nu\} \quad \text{and} \quad \hat{S}_k = \{\hat{y} = \{y,m,n\} \in \hat{E} \colon y \in S_k\}.$$

By the above considerations, for each $k \in \mathrm{N}$ the eigenvector $\hat{y}_k = \{\, y_k,\ m_k,\ r$ corresponding to the eigenvalue λ_k of the operator $L,$ belongs to the class $\hat{S}_k$; the sets $\hat{S}_k^\nu \, (\hat{S}_k)$ are disjoint and are open subsets of the Banach space $\hat{E}.$ Moreover, if $\hat{y} = \{y,m,n\} \in \partial \hat{S}_k^\nu \, (\partial \hat{S}_k),$ then there exists $\xi \in [0,1]$ such that $y(\xi) = y'(\xi) = y''(\xi) = y'''(\xi) = 0.$

Lemma 3. *Let* $(\lambda, y) \in R \times E$ *be a solution to the nonlinear eigenvalue problem (1)-(4) satisfying the condition* $y \in \partial S_k^\nu \, (\partial S_k).$ *Then* $y \equiv 0.$

Corollary 1. *Let* $(\lambda, \hat{y}) \in R \times \hat{E}$ *be a solution to the nonlinear eigenvalue problem (11) satisfying the condition* $y \in \partial \hat{S}_k^\nu \, (\partial \hat{S}_k).$ *Then* $\hat{y} \equiv \hat{0},$ *where* $\hat{0} = \{0,0,0\}.$

4. Global bifurcation from zero and infinity of solutions to problem (1)-(4)

Let $\delta > 0$ be an arbitrary fixed sufficiently small number. Since $\lambda_1 = 0$ is eigenvalue of operator L consider the following approximate problem

$$L\hat{y} + \delta J_3 \hat{y} = \lambda \hat{y} + F(\lambda, \hat{y}) + G(\lambda, \hat{y}),\ \hat{y} \in D(L), \tag{27}$$

where the operator $J_3 : \hat{E} \oplus R^2 \to \hat{E} \oplus R^2$ is defined as follows:

$$J_3 \hat{y} = J_3\{y, m, n\} = \{0, 0, |c|^{-1} n\}.$$

Since $c < 0$ we have

$$(J_3 \hat{y}, \hat{y})_H = |c|^{-2} n^2 \geq 0,$$

and consequently, the operator $L_\delta = L + \delta J_3$ is positive. Therefore, all eigenvalues of this operator are positive. Moreover, by the arguments in the proof of [9, Theorem 4.5] we can show that the eigenvalues of the operator L_δ are simple and form infinitely increasing sequence $\{\lambda_k(\delta)\}_{k=1}^{\infty}$.

If we multiply both sides of the equation

$$L_\delta \hat{y} = \lambda \hat{y},\ \hat{y} \in D(L),$$

by $\hat{y}$ scalarly in the space $H,$ then we get

$$\lambda = \frac{(L_\delta \hat{y}, \hat{y})_H}{(\hat{y}, \hat{y})_H} = \frac{(L\hat{y}, \hat{y})_H}{(\hat{y}, \hat{y})_H} + \delta \frac{(J_3 \hat{y}, \hat{y})_H}{(\hat{y}, \hat{y})_H} = \frac{(L\hat{y}, \hat{y})_H}{(\hat{y}, \hat{y})_H} + \frac{y^2(1)}{\int\limits_0^1 y^2(x)dx + ay^2(0) - cy^2(1)}.$$

Hence by maximal-minimal property of eigenvalues [1, formula (22)] it follows from last relation that

$$\lambda_k \leq \lambda_k(\delta) \leq \lambda_k + \delta |c|^{-1},$$

which implies that

$$\lambda_k(\delta) \to \lambda_k \text{ as } \delta \to 0+. \tag{28}$$

Since the operator L_δ has a compact resolvent, the operator

$$L_\delta^{-1} : C[0,1] \oplus R^2 \to D(L)$$

exists and is a compact (completely continuous) operator. Then problem (27) is reduced to the following nonlinear eigenvalue problem

$$\hat{y} = \lambda \hat{y} + F_\delta(\lambda, \hat{y}) + G_\delta(\lambda, \hat{y}) \tag{29}$$

with completely continuous operators $\hat{L}_\delta$, $\hat{F}_\delta$ and $\hat{G}_\delta$ which are defined as follows:

$$\hat{L}_\delta = L_\delta^{-1},\ \hat{F}_\delta = L_\delta^{-1} F : R \times \hat{E} \to \hat{E} \quad \text{and} \quad \hat{G}_\delta = L_\delta^{-1} G : R \times \hat{E} \to \hat{E}.$$

Lemma 4. Let condition (6) be satisfied. Then for any bounded interval $\Lambda \subset R,$

$$\hat{G}_\delta(\lambda, \hat{y}) = o(\|\hat{y}\|_3) \text{ as } \|\hat{y}\|_3 \to 0, \tag{30}$$

uniformly for $\lambda \in R.$

Lemma 5. Let condition (7) be satisfied. Then for any bounded interval $\Lambda \subset R$,

$$\hat{G}_\delta(\lambda, \hat{y}) = o(\|\hat{y}\|_3) \text{ as } \|\hat{y}\|_3 \to +\infty, \tag{31}$$

uniformly for $\lambda \in R$.

The proof of Lemma 4 is obvious, and the proof of Lemma 5 is based on Lemma 5.5 of [8].

Let in (1) $f \equiv 0$. Then (29) takes the following form:

$$\hat{y} = \lambda \hat{L}_\delta \hat{y} + \hat{G}_\delta(\lambda, \hat{y}). \tag{32}$$

If condition (6) is satisfied, then using Corollary 1, Lemma 4, [18, Lemma 1.24] and [14, Theorem 2] for problem (32) we have the following global bifurcation results.

Theorem 2. *Let condition (6) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+, -\}$ *there exists a continuum* $\hat{C}_{k,\delta}^\nu$ *of solutions of problem (32) which meets* $(\lambda_k(\delta), \hat{0})$, *is contained in* $(R \times \hat{S}_k^\nu) \bigcup \{(\lambda_k(\delta), \hat{0})\}$ *and is unbounded in* $R \times \hat{E}$.

By $\Omega \subset R$ we denote of a bounded open neighbourhood of the point $(\lambda_k, 0)$ such that $(\lambda_j, 0) \notin \Omega$ for $j \neq k$. It is obvious that for any sufficiently small δ the eigenvalues $\lambda_k(\delta)$ of the operator L_δ lies in Ω and by Theorem 2 there exists $(\lambda_{k,\delta}^\nu, y_{k,\delta}^\nu) \in C_{k,\delta}^\nu \bigcap \Omega$. Since problem (32) is equivalent to the problem

$$\begin{cases} \ell(y) = \lambda y + g(x, y, y', y'', y''', \lambda), \; 0 < x < 1, \\ y''(0) = y''(1) = Ty(0) - a\lambda y(0) = 0, \\ Ty(1) - (c\lambda + \delta) y(1) = 0, \end{cases} \tag{33}$$

it follows that the set $\{y_{k,\delta}^\nu\}$ is bounded in $C^4[0,1]$. Let $\hat{w}_{k,\delta}^\nu = \dfrac{\hat{y}_{k,\delta}^\nu}{|y_{k,\delta}^\nu|_3}$. If $|y_{k,\delta}^\nu|_3 \to 0$ as $\delta \to 0$ then from the relations

$$\begin{cases} \ell(w_{k,\delta}^\nu) = \lambda_{k,\delta}^\nu w_{k,\delta}^\nu + g(x, y_{k,\delta}^\nu, y_{k,\delta}'^\nu, y_{k,\delta}''^\nu, y_{k,\delta}'''^\nu, \lambda) \big/ |y_{k,\delta}^\nu|_3, \; 0 < x < 1, \\ w_{k,\delta}''^\nu(0) = w_{k,\delta}''^\nu(1) = 0, \; Tw_{k,\delta}^\nu(0) - a\lambda_{k,\delta}^\nu w_{k,\delta}^\nu(0) = 0, \\ Tw_{k,\delta}^\nu(1) - (c\lambda_{k,\delta}^\nu + \delta) w_{k,\delta}^\nu(1) = 0, \end{cases} \tag{34}$$

we obtain that $\{w_{k,\delta}^\nu\}$ is bounded in $C^4[0,1]$. Then by the Arzela-Ascoli theorem we can assume that $\{(\lambda_{k,\delta}^\nu, w_{k,\delta}^\nu)\}$ converges to some (λ, w) in $R \times E$ as $\delta \to 0$. Hence passing to limit in (34) as $\delta \to 0$ we obtain

$$\begin{cases} \ell(w) = \lambda w, \ 0 < x < 1, \\ w''(0) = w''(1) = 0, Tw(0) - a\lambda w(0) = 0, \ Tw(1) - c\lambda) w(1) = 0, \end{cases}$$

which by $w_{k,\delta}^{\nu} \in S_k^{\nu}$ implies that $\lambda = \lambda_k$. Therefore, $\{(\lambda_{k,\delta}^{\nu}, y_{k,\delta}^{\nu})\}$ converges to some $(\lambda_k, 0)$ in $R \times E$ as $\delta \to 0$. Since $\{(\lambda_{k,\delta}^{\nu}, y_{k,\delta}^{\nu})\} \subset \partial\Omega$, it follows that $(\lambda_k, 0) \in \partial\Omega$, which contradicts the fact that Ω is a bounded open neighbourhood of $(\lambda_k, 0)$. Thus $(\lambda_{k,\delta}^{\nu}, y_{k,\delta}^{\nu}) \to (\lambda, \vartheta)$, $\vartheta \neq 0$, in $R \times E$ as $\delta \to 0$, where $(\lambda, \vartheta) \in \partial\Omega$ and by (33) (λ, ϑ) is solution of problem (1)-(4) with $f \equiv 0$, and consequently, by (32) $(\lambda, \hat{\vartheta})$ is a nontrivial solution of problem

$$L\hat{y} = \lambda \hat{y} + G(\lambda, \hat{y}). \tag{35}$$

Since this argument is true for every such Ω, it follows from an elementary argument from point set topology (see, for example, the proof of [17, Corollary 1.34] and [19, Theorem 2.4]) that problem (35) has a continuum C_k^{ν} as in Theorem 2.

Thus we obtain the following result (In fact, we complete the proof of [16, Theorem 3]).

Theorem 3. *Let condition (6) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+, -\}$ *there exists a continuum* $\hat{C}_k^{\nu}$ *of solutions of problem (35) which meets* $(\lambda_k, \hat{0})$, *is contained in* $(R \times \hat{S}_k^{\nu}) \bigcup \{(\lambda_k, \hat{0})\}$ *and is unbounded in* $R \times \hat{E}$ *(in this case either* $\hat{C}_k^{\nu}$ *meets* (λ, ∞) *for some* $\lambda \in R$, *or the natural projection of* $\hat{C}_k^{\nu}$ *onto* $\{(\lambda, \hat{0}) \colon \lambda \in R\}$ *is unbounded).*

Since there is a one-to-one correspondence (12) between the solutions of problem (1)-(4) for $f \equiv 0$ and problem (35), the following result follows from Theorem 3.

Theorem 4. *Let* $f \equiv 0$ *and condition (6) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+, -\}$ *there exists a continuum* C_k^{ν} *of solutions of problem (1)-(4) which meets* $(\lambda_k, 0)$, *is contained in* $(R \times S_k^{\nu}) \bigcup \{(\lambda_k, 0)\}$ *and is unbounded in* $R \times E$ *(in this case either* C_k^{ν} *meets* (λ, ∞) *for some* $\lambda \in R$, *or the natural projection of* C_k^{ν} *onto* $\{(\lambda, 0) \colon \lambda \in R\}$ *is unbounded).*

Let condition (7) holds. If $(\lambda,\hat{y})\in R\times\hat{E}$ is a nontrivial solution of problem (11) then dividing both sides (11) by $\|\hat{y}\|_3^2$ and denoting $\hat{w}=\dfrac{\hat{y}}{\|\hat{y}\|_3^2}$ we get

$$L\hat{w}=\lambda\hat{w}+\frac{F(\lambda,\hat{y})}{\|\hat{y}\|_3^2}+\frac{G(\lambda,\hat{y})}{\|\hat{y}\|_3^2}. \tag{36}$$

We have the following relations:

$$\|\hat{w}\|_3=\frac{1}{\|\hat{y}\|_3},\quad \|\hat{y}\|_3=\frac{1}{\|\hat{w}\|_3}\quad\text{and}\quad \hat{y}=\|\hat{y}\|_3^2\,\hat{w}=\frac{\hat{w}}{\|\hat{w}\|_3^2}.$$

Then (36) can be rewritten as follows:

$$L\hat{w}=\lambda\hat{w}+\|\hat{w}\|_3^2\;F\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_3^2}\right)+\|\hat{w}\|_3^2\;G\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_3^2}\right). \tag{37}$$

Let

$$K(\lambda,\hat{w})=\|\hat{w}\|_3^2\;F\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_3^2}\right)\ \text{and}\ \ H(\lambda,\hat{w})=\|\hat{w}\|_3^2\;G\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_3^2}\right),\ \hat{w}\neq\hat{0}.$$

By condition (5) for any $x\in[0,1]$ and $\lambda\in R$ we have

$$|\,f(x,y(x),y'(x),y''(x),y'''(x),\lambda)\,|\le M\,|\,y\,|\le M\,|\,y\,|_0\le M\,|\,y\,|_3\le M\,\|\,\hat{y}\,\|_3 \tag{38}$$

Then it follows from (38) that for any $\lambda\in R$ the following relation holds:

$$\|F(\lambda,\hat{y})\|_0=|\,f(x,y(x),y'(x),y''(x),y'''(x),\lambda)\,|_0\le M\,\|\,y\,\|_3, \tag{39}$$

which implies that

$$\|K(\lambda,\hat{w})\|=\|\hat{w}\|_3^2\left\|F\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_3^2}\right)\right\|_0=\|\hat{w}\|_3^2\|F(\lambda,\hat{y})\|_0\le$$

$$\le M\,\|\hat{w}\|_3^2\,\|\hat{y}\|_3=M\,\|\hat{w}\|_3 \tag{40}$$

It is obvious that

$$\|\hat{y}\|_3\to+\infty\Leftrightarrow\|\hat{w}\|_3\to 0. \tag{41}$$

By (30) for any bounded interval $\Lambda\subset R$,

$$\frac{\|G(\lambda,\hat{y})\|_0}{\|\hat{y}\|_3}\to 0\quad\text{as}\quad\|\hat{y}\|_3\to+\infty, \tag{42}$$

uniformly for $\lambda \in \Lambda.$ Then by (41) it follows from (42) that

$$\frac{\| H(\lambda, \hat{w}) \|_0}{\| \hat{w} \|_3} = \| \hat{w} \|_3 \left\| G\left(\lambda, \frac{\hat{w}}{\|\hat{w}\|_3^2} \right) \right\|_0 = \frac{\|G(\lambda, \hat{y})\|_0}{\| \hat{y} \|_3} \to 0 \text{ as } \| \hat{w} \|_3 \to 0 \quad (43)$$

uniformly for $\lambda \in \Lambda.$ We can be extend H to $\hat{w} = \hat{0}$ by setting $H(\lambda, \hat{0}) = \hat{0}.$ Therefore, the operator $H : R \times \hat{E} \to \hat{E}$ is continuous. Using the proof of [19, Theorem 2.4], we can show that the operator H is compact, and hence this operator is completely continuous.

Thus by (42) and (43) the bifurcation from infinity problem (11) under transformation

$$(\lambda, \hat{y}) \to (\lambda, \hat{w}) = \left(\lambda, \frac{\hat{y}}{\|\hat{y}\|_3^2} \right) \quad (44)$$

is reduced to the bifurcation from zero problem

$$L\hat{w} = \lambda \hat{w} + K(\lambda, \hat{w}) + H(\lambda, \hat{w}). \quad (45)$$

Let $\hat{C}$ and $\hat{D}$ be the closures of the sets of non-trivial solutions of problems (11) and (45), respectively. Note that transformation (44) maps $\hat{C}$ into $\hat{D}$ and, heuristically, interchanges points at $\hat{y} = \hat{0}$ (respectively, $\hat{y} = \infty$) with points at $\hat{y} = \infty$ (respectively, $\hat{y} = \hat{0}$).

Remark 6. If only condition (7) is satisfied, then we cannot claim that the statements of Lemma 3 and Corollary 1 are true.

It is obvious that if $f \equiv 0,$ then the transformation (44) transforms the bifurcation from infinity problem (35) into the following the bifurcation from zero problem

$$L\hat{w} = \lambda \hat{w} + H(\lambda, \hat{w}). \quad (46)$$

Then, by Remark 6, [14, Theorem 2], from Theorem 3 we obtain the following global bifurcation results for problem (46) and (1)-(3) for $f \equiv 0,$ respectively.

Theorem 5. *Let condition (7) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+, -\}$ *there exists a continuum* $\hat{\Phi}_k^\nu$ *of solutions of problem (46) emanating* *from* $(\lambda_k, \hat{0})$ *and a neighbourhood* $\hat{Q}_k$ *of* $(\lambda_k, \hat{0})$ *such that (i)* $(\hat{\Phi}_k^\nu \setminus \{(\lambda_k, \hat{0})\}) \cap$ $\cap \hat{Q}_k \subset R \times \hat{S}_k^\nu$; *(ii) either* $\hat{\Phi}_k^\nu$ *meets* $(\lambda_{k'}, \hat{0})$ *for some* $(k', \nu') \neq (k, \nu)$ *with the set* $R \times S_{k'}^{\nu'}$, *or* $\hat{\Phi}_k^\nu$ *meets* $(\lambda, \hat{0})$ *for some* $\lambda \in R$, *or the natural projection of* $\hat{\Phi}_k^\nu$ *onto* $\{(\lambda, \hat{0}) \colon \lambda \in R\}$ *is unbounded.*

Theorem 6. *Let condition (7) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\nu \in \{+, -\}$ *there exists a continuum* Φ_k^ν *of solutions of problem (1)-(3) for* $f \equiv 0$ *emanating from* $(\lambda_k, 0)$ *and a neighbourhood* Q_k *of* $(\lambda_k, 0)$ *such that (i)* $(\Phi_k^\nu \setminus \{(\lambda_k, 0)\}) \cap \ \cap Q_k \subset R \times S_k^\nu$; *(ii) either* Φ_k^ν *meets* $(\lambda_{k'}, 0)$ *for some* $(k', \nu') \neq (k, \nu)$ *with the set* $R \times S_{k'}^{\nu'}$, *or* Φ_k^ν *meets* $(\lambda, 0)$ *for some* $\lambda \in R$, *or the natural projection of* Φ_k^ν *onto* $\{(\lambda, 0) \colon \lambda \in R\}$ *is unbounded.*

Using the proof [20, Theorem 3.3] from Theorems 3 and 5 we obtain the following results.

Theorem 7. *Let both conditions (6) and (7) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\{+, -\}$ *the relation* $\hat{\Phi}_k^\nu \subset R \times \hat{S}_k^\nu$ *holds (in this case the first alternative of the statement (ii) of Theorem 5 cannot hold). Moreover, if* $\hat{\Phi}_k^\nu$ *meets* $(\lambda, \hat{0})$ *for some* $\lambda \in R$, *then* $\lambda = \lambda_k$, *and if* $\hat{C}_k^\nu$ *meets* (λ, ∞) *for some* $\lambda \in R$, *then* $\lambda = \lambda_k$.

Theorem 8. *Let both conditions (6) and (7) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\{+, -\}$ *the relation* $\Phi_k^\nu \subset R \times S_k^\nu$ *holds (in this case the first alternative of the statement (ii) of Theorem 6 cannot hold). Moreover, if* Φ_k^ν *meets* $(\lambda, \hat{0})$ *for some* $\lambda \in R$, *then* $\lambda = \lambda_k$, *and if* C_k^ν *meets* (λ, ∞) *for some* $\lambda \in R$, *then* $\lambda = \lambda_k$.

Now we consider problem (1)-(4) in the general case, i.e., in the case when f is not identically equal to zero.

Let

$$J_k = [\lambda_k - M, \lambda_k + M], \ k \in \mathrm{N}.$$

Lemma 6. *Let condition (6) be satisfied. Then for each* $k \in \mathrm{N}$ *and each* $\{+, -\}$ *the set of bifurcation points of problem (11) with respect to the set* $R \times \hat{S}_k^\nu$ *is nonempty. Moreover, if* $(\lambda, \hat{0})$ *is a bifurcation point of this problem with respect to* *the set* $R\times\hat{S}_k^{\nu}$*, then* $\lambda\in J_k$.

For each $k\in\mathrm{N}$ and each $\{+,-\}$ by $\hat{D}_{k,\lambda}^{\nu}$ denote the component of $\hat{C}$ which bifurcating from the bifurcation point $(\lambda,\hat{0})$ of problem (11) with respect to the set $R\times\hat{S}_k^{\nu}$.Let

$$\hat{D}_k^{\nu}=\bigcup_{\lambda\in J_k}\hat{D}_{k,\lambda}^{\nu}\ \text{ and }\ \tilde{\hat{D}}_k^{\nu}=\hat{D}_k^{\nu}\bigcup(J_k\times\{0\}).$$

It is obvious that the set $\tilde{\hat{D}}_k^{\nu}$ is connected in $R\times\hat{E}$, but $\hat{D}_k^{\nu}$ may not be connect $R\times\hat{E}$.

Theorem 9. *Let condition (6) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *the set* $\hat{D}_k^{\nu}$ *is nonempty, contained in* $(R\times\hat{S}_k^{\nu})\bigcup(J_k\times\{\hat{0}\})$ *and is unbounded in* $R\times\hat{E}$ *(in this case either* $\hat{D}_k^{\nu}$ *meets* (λ,∞) *for some* $\lambda\in R$*, or the natural projection of* $\hat{D}_k^{\nu}$ *onto* $\{(\lambda,\hat{0}):\lambda\in R\}$ *is unbounded).*

Let

$$C=\{(\lambda,y)\in R\times E:(\lambda,\hat{y})\in\hat{C}\}\ \text{ and }\ D_k^{\nu}=\{(\lambda,y)\in R\times E:(\lambda,\hat{y})\in\hat{D}_k^{\nu}\}.$$

Then in view of (12) by Theorem 9 we have the following result.

Theorem 10. *Let condition (6) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *the set* D_k^{ν} *is nonempty, contained in* $(R\times S_k^{\nu})\bigcup(J_k\times\{0\})$ *and is unbounded in* $R\times E$ *(in this case either* D_k^{ν} *meets* (λ,∞) *for some* $\lambda\in R$*, or the natural projection of* D_k^{ν} *onto* $\{(\lambda,0):\lambda\in R\}$ *is unbounded).*

Taking into account Remarks 6 and Theorem 5, by applying the transformation (44) from Lemma 6 and Theorem 9, we obtain the following results.

Lemma 7. *Let condition (7) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\{+,-\}$ *the set of asymptotic bifurcation points of problem (11) with respect to the set* $R\times\hat{S}_k^{\nu}$ *is nonempty. Moreover, if* (λ,∞) *is a bifurcation point of this problem with respect to the set* $R\times\hat{S}_k^{\nu}$*, then* $\lambda\in J_k$.

For each $k \in \mathrm{N}$ and each $\{+,-\}$ by $\hat{\Psi}^{\nu}_{k,\lambda}$ denote the component of $\hat{C}$ which bifurcating from the asymptotic bifurcation point (λ,∞) of problem (11) with respect to the set $R\times\hat{S}^{\nu}_{k}$.Let

$$\hat{\Psi}^{\nu}_{k} = \bigcup_{\lambda\in J_k} \hat{\Psi}^{\nu}_{k,\lambda} \quad \text{and} \quad \tilde{\hat{\Psi}}^{\nu}_{k} = \hat{\Psi}^{\nu}_{k} \bigcup (J_k \times\{0\}).$$

It is obvious that the set $\tilde{\hat{\Psi}}^{\nu}_{k}$ is connected in $R\times\hat{E},$ but $\hat{\Psi}^{\nu}_{k}$ may not be connect $R\times\hat{E}.$

Theorem 11. *Let condition (7) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *the set* $\hat{\Psi}^{\nu}_{k}$ *is nonempty and either (i)* $\hat{\Psi}^{\nu}_{k}$ *meets* $(\lambda_{k'},\infty)$ *for some* $(k',\nu')\neq(k,\nu)$ *with the set* $R\times\hat{S}^{\nu'}_{k'}$*, or (ii)* $\hat{\Psi}^{\nu}_{k}$ *meets* $(\lambda,\hat{0})$ *for some* $\lambda\in R,$ *or (iii) the natural projection of* $\hat{\Psi}^{\nu}_{k}$ *onto* $\{(\lambda,\hat{0}):\lambda\in R\}$ *is unbounded.*

Let $\Psi^{\nu}_{k} = \{(\lambda,y)\in R\times E:(\lambda,\hat{y})\in\hat{\Psi}^{\nu}_{k}\}$. Then by (12) from Theorem 1 1 we get the following result.

Theorem 12. *Let condition (7) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\nu\in\{+,-\}$ *the set* $\hat{\Psi}^{\nu}_{k}$ *is nonempty and either (i)* Ψ^{ν}_{k} *meets* $(\lambda_{k'},\infty)$ *for some* $(k',\nu')\neq(k,\nu)$ *with the set* $R\times S^{\nu'}_{k'}$*, or (ii)* Ψ^{ν}_{k} *meets* $(\lambda,0)$ *for some* $\lambda\in R,$ *or (iii) the natural projection of* Ψ^{ν}_{k} *onto* $\{(\lambda,0):\lambda\in R\}$ *is unbounded.*

Using the proof [20, Theorem 3.3] from Theorems 9 and 11 we obtain the following results.

Theorem 7. *Let both conditions (6) and (7) be satisfied. Then for each* $k\in\mathrm{N}$ *and each* $\{+,-\}$ *the relation* $\hat{\Psi}^{\nu}_{k}\subset R\times\hat{S}^{\nu}_{k}$ *holds (in this case the alternative (i) of Theorem 11 cannot hold). Moreover, if* $\hat{\Psi}^{\nu}_{k}$ *meets* $(\lambda,\hat{0})$ *for some* $\lambda\in R,$ *then* $\lambda\in J_k$*, and if* $\hat{D}^{\nu}_{k}$ *meets* (λ,∞) *for some* $\lambda\in R,$ *then* $\lambda\in J_k$.

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