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THE PROBLEM OF OPTIMAL CONTROL BY THE LEADING COEFFICIENT OF THE SECOND ORDER HYPERBOLIC EQUATION WITH DISCONTINIOUS SOLUTION

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Abstract

In this work, the problem of finding the leading coefficient of the second order hyperbolic equation with a discontinuous solution is studied. The considered problem is reduced to the optimal control problem. The existence of the optimal pair is proved, the convergence of the adapted penalty method is shown, and a necessary condition for optimality in the form of the variational inequality is derived.

Keywords: truncated solution, prime factor, optimal control, penalty method, necessary condition for optimality.

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1.Introduction

Inverse problems for partial differential equations are studied by various methods, for example, regularization method, quasi-inversion method, quasi-

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solution method, as well [3, 4, 9]. One of these methods is variational method or optimization method [3, 4, 6]. Applying this method the considered problem is reduced to the problem of finding the minimum of the constructed functional with the help of additional information, and the obtained problem is studied with the help of the methods of optimal control theory [1, 2, 8]. In this paper, the problem of finding the leading coefficient of a two-order hyperbolic equation with a discontinuous solution is studied. Thus, the functional is constructed using the characteristics of the problem under consideration. The existence of the optimal pair that gives a minimum to this functional is proved [7]. Then, a new functional adapted to this optimal pair is constructed, the convergence of the adapted penalty method is proved, and a necessary condition for optimality is derived in the form of a variational inequality.

2. **Problem formulation**

Let in the cylinder $Q=\Omega\times(0,T)$ the controlled problem is described by the following hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial u}{\partial x_i}\right)-u^3=f(x,t),\ (x,t)\in Q\ , \tag{1}$$

and initial and boundary conditions

$$u(x,0)=u_0(x),\quad \frac{\partial u}{\partial t}(x,0)=u_1(x),\ x\in\Omega, \tag{2}$$

$$u\big|_{\Sigma}=0. \tag{3}$$

Here Ω is a bounded domain from R^n ($n\le 3$) with smooth boundary Γ; $T>0$ is a given positive number; $\Sigma=\Gamma\times(0,T)$ is a lateral surface of the cylinder Q; $f(x,t)\in L_2(Q)$, $u_0(x)\in \overset{0}{W}{}_2^1(\Omega)$, $u_1(x)\in L_2(\Omega)$ are given functions.

Let the sought control $\upsilon(x)$ belongs to the class

$$V=\left\{\upsilon(x):\upsilon(x)\in \overset{0}{W}{}_2^1(\Omega),\ \ \nu_0\le\upsilon(x)\le\mu_0,\ \ \left|\frac{\partial\upsilon(x)}{\partial x_i}\right|\le\mu_i\ a.e.\ in\ \Omega,\ \ i=\overline{1,n}\right\}, \tag{4}$$

where ν_0, μ_0, $\mu_i, i=\overline{1,n}$ are given positive numbers.

If $\upsilon \in V$, $u \in L_6(Q)$ then as we get from (1)-(3) the solution $u = u(x, t)$ from the class

$$U = \left\{ u : u \in L_\infty(0, T; \overset{0}{W}{}_2^1(\Omega)), \frac{\partial u}{\partial t} \in L_\infty(0, T; L_2(\Omega)) \right\},$$

satisfies the condition $u(x, 0) = u_0(x)$ and integral identity

$$\begin{aligned} \int_Q \left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \upsilon(x) \frac{\partial u}{\partial x_i} \cdot \frac{\partial \eta}{\partial x_i} \right] dxdt - \int_Q u^3 \eta dxdt = \\ = \int_Q f \eta dxdt + \int_\Omega u_1(x) \eta(x, 0) dx \end{aligned}, \quad (5)$$

for $\forall \eta(x, t) \in C^1(\overline{Q})$, $\eta(x,T) = 0$. Such pair $\{\upsilon, u\}$ is called to be an admissible pair.

We suppose that the set of the admissible pairs is not empty, i.e. $\{\upsilon, u\} \neq \varnothing$.

Consider the problem of minimizing the following functional in the set of possible pairs

$$J(\upsilon, u) = \frac{1}{6}\|u - u_d\|_{L_6(Q)}^6 + \frac{N}{2}\|\upsilon\|_{W_2^1(\Omega)}^2, \quad (6)$$

where $u_d \in L_6(Q)$ is a given function and $N > 0$ is a given positive number.

3.Existence of the optimal pair in problem (1)-(4), (6)

Let us now show the existence of an optimal pair for the considered optimal control problem.

Theorem 1. Let the conditions imposed on the data of problem (1)-(4), (6) be satisfied. Then there exists an optimal pair $\{\tilde{\upsilon}, \tilde{u}\}$ in this problem i.e.

$$J(\tilde{\upsilon}, \tilde{u}) = \inf_{\{\upsilon, u\}} J(\upsilon, u), \quad (7)$$

where $\{\upsilon, u\}$ are admissible pairs.

Proof. Let $\{\upsilon_k, u_k\}$ be a minimizing sequence

$$\lim_{k \to \infty} J(\upsilon_k, u_k) = \inf_{\{\upsilon, u\}} J(\upsilon, u). \quad (8)$$

From the definition of functional (6) we obtain

$$\|\upsilon_k\|_{W_2^1(\Omega)} + \|u_k\|_{L_6(Q)} \leq c. \quad (9)$$

Thus, using relations

$$\frac{\partial^2 u_k}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\upsilon_k(x) \frac{\partial u_k}{\partial x_i} \right) = u_k^{\ 3} + f \ , \tag{10}$$

$$u_k(x,\,0) = u_0(x), \quad \frac{\partial u_k}{\partial t}(x,\,0) = u_1(x)\,, \;\; x \in \Omega,\; u_k\big|_{\Sigma} = 0\,, \tag{11}$$

and results from [5. pp. 209-215] we obtain that

$$\left\| u_k \right\|_{L_\infty(0,\,T;\,\overset{0}{W}{}_2^1(\Omega))} + \left\| \frac{\partial u_k}{\partial t} \right\|_{L_\infty(0,\,T;\,L_2(\Omega))} \le c\ . \tag{12}$$

Here and later we will denote by c various constants independent of estimated quantities and controls.

Then from the sequence $\{\upsilon_k, u_k\}$ one may extract a subsequence (for the sake of simplicity we'll denote it also as $\{\upsilon_k, u_k\}$) for which

$$\upsilon_k \to \tilde{\upsilon} \quad \text{weak in } W_2^1(\Omega)\,,$$

$$u^k \to \tilde{u} \quad *\text{-weak in } L_\infty(0,T;\, W_2^1(\Omega))\,, \tag{13}$$

$$\frac{\partial u_k}{\partial t} \to \frac{\partial \tilde{u}}{\partial t} \quad *\text{-weak in } L_\infty(0,T;\, L_2(\Omega))$$

are valid as $k \to \infty$. From the embedding theorems [5. pp. 84-85] we obtain that as $k \to \infty$

$$\upsilon_k \to \tilde{\upsilon} \quad \text{strong in } L_2(\Omega)\,, \tag{14}$$

$$u_k \to \tilde{u} \quad \text{weak in } L_6(Q)\text{, for } p < 6 \text{ strong in } L_p(Q) \text{ and a.e. in } Q\ . \tag{15}$$

Let us write the condition

$$u_k(x,\,0) = u_0(x) \tag{16}$$

and integral identity (5) for $\upsilon = \upsilon_k(x)$, $u = u_k(x,\,t)$

$$\begin{aligned} \int_Q \left[-\frac{\partial u_k}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \upsilon_k(x) \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial \eta}{\partial x_i} \right] dxdt - \int_Q u_k^{\ 3} \eta dxdt = \\ = \int_Q f\eta dxdt + \int_\Omega u_1(x)\eta(x,\,0)dx \end{aligned} \tag{17}$$

$\forall \eta(x,\,t) \in C^1(\overline{Q}), \quad \eta(x,T) = 0$.

Considering relations (13), (14) and (15) we can pass to limit in (16) and (17) as $k \to \infty$. Then $\{\tilde{\upsilon},\, \tilde{u}\}$ will be an admissible pair.

Since the functional $J(\upsilon, u)$ is weakly semi-continuous we get

$$\underline{\lim}_{k\to\infty} J(\upsilon_k, u_k) \ge J(\tilde{\upsilon}, \tilde{u}) . \tag{18}$$

The (8) and (18) implies

$$\inf_{\{\upsilon,\, u\}} J(\upsilon, u) = J(\tilde{\upsilon}, \tilde{u}) .$$

Therefore $\{\tilde{\upsilon}, \tilde{u}\}$ is an optimal pair. Theorem is proved.

4.Convergence of the adopted penalty method

Let us write the adopted functional

$$J_\varepsilon^a(\upsilon, u) = \frac{1}{6}\|u - u_d\|_{L_6(Q)}^6 + \frac{N}{2}\|\upsilon\|_{W_2^1(\Omega)}^2 + \\ + \frac{1}{2\varepsilon}\left\|\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial u}{\partial x_i}\right) - u^3 - f\right\|_{L_2(Q)}^2 + \\ + \frac{1}{2}\|u - \tilde{u}\|_{L_2(Q)}^2 + \frac{1}{2}\|\upsilon - \tilde{\upsilon}\|_{W_2^1(\Omega)}^2 \tag{19}$$

for the optimal pair $\{\tilde{\upsilon}, \tilde{u}\}$. Here the functions $\upsilon,\ u$ satisfy the conditions

$$\upsilon \in V\ ,\ \ u \in L_6(Q),$$

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial u}{\partial x_i}\right) \in L_2(Q) \tag{20}$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x)\ ,\quad x \in \Omega,\ u\big|_\Sigma = 0.$$

Now we investigate the problem of minimization of the functional $J_\varepsilon^a(\upsilon, u)$ subject to conditions (20). One can show the existence of the optimal pair $\{\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon\}$ for this problem similar to Theorem 1.

Theorem 2. Let the pair $\{\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon\}$ be a solution to the problem $J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) = \inf\ J_\varepsilon^a(\upsilon, u)$ for each $\varepsilon > 0$. Then for $\varepsilon \to 0$ we can write

$$\tilde{\upsilon}_\varepsilon \to \tilde{\upsilon} \text{ strong in } W_2^1(\Omega), \tag{21}$$

$$\tilde{u}_\varepsilon \to \tilde{u} \text{ strong in } L_6(Q). \tag{22}$$

Here $\{\tilde{\upsilon}, \tilde{u}\}$ is a selected optimal pair.

Proof. It is clear that

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) = \inf J_\varepsilon^a(\upsilon, u) \le J_\varepsilon^a(\tilde{\upsilon}, \tilde{u}) = J(\tilde{\upsilon}, \tilde{u}) . \tag{23}$$

From this due to the definition of the functional we get

$$\|\tilde{\upsilon}_\varepsilon\|_{W_2^1(\Omega)} + \|\tilde{u}_\varepsilon\|_{L_6(Q)} \le c , \tag{24}$$

where c is a constant independent from ε. The following relations also hold true

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\upsilon_\varepsilon(x)\frac{\partial u_\varepsilon}{\partial x_i}\right) - u_\varepsilon^{\,3} = \sqrt{\varepsilon} g_\varepsilon + f , \tag{25}$$

$$\tilde{u}_\varepsilon(x, 0) = u_0(x) , \ \frac{\partial \tilde{u}_\varepsilon}{\partial t}(x,0) = u_1(x) , \ \tilde{u}_\varepsilon\big|_\Sigma = 0 , \tag{26}$$

where the function $g_\varepsilon(x, t)$ satisfies $\|g_\varepsilon\|_{L_2(Q)} \le c$.

From (24), (25) and (26) follows

$$\|\tilde{u}_\varepsilon\|_{L_\infty(0, T; \overset{0}{W}{}_2^1(\Omega))} \le c \ , \ \left\|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right\|_{L_\infty(0, T; L_2(\Omega))} \le c . \tag{27}$$

Considering the embedding theorem $W_2^1(\Omega) \subset L_2(\Omega)$ [5] from the sequence $\{\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon\}$ one can extract a subsequence (which will be denoted as same) as $\varepsilon \to 0$

$$\tilde{\upsilon}_\varepsilon \to \tilde{\tilde{\upsilon}} \text{ strong in } L_2(\Omega) , \tag{28}$$

$$\tilde{u}_\varepsilon \to \tilde{\tilde{u}} \ \ *\text{-weak in } L_\infty(0,T; \overset{0}{W}{}_2^1(\Omega)) ,$$

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t} \to \frac{\partial \tilde{\tilde{u}}}{\partial t} \ \ *\text{-weak in } L_\infty(0,T; L_2(\Omega)) ,$$

$$\tilde{u}_\varepsilon \to \tilde{\tilde{u}} \text{ weak in } L_6(Q) .$$

Thus the following relations hold true in the weak sense

$$\frac{\partial^2 \tilde{\tilde{u}}_\varepsilon}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\tilde{\upsilon}}_\varepsilon(x)\frac{\partial \tilde{\tilde{u}}_\varepsilon}{\partial x_i}\right) - \tilde{\tilde{u}}_\varepsilon^{\,3} = f(x,t) ,$$

$$\tilde{\tilde{u}}(x, 0) = u_0(x) , \ \frac{\partial \tilde{\tilde{u}}(x, 0)}{\partial t} = u_1(x) , \ \tilde{\tilde{u}}\big|_\Sigma = 0 .$$

Therefore, from the inequality

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) + \frac{1}{2}\left\|\tilde{u} - \tilde{\tilde{u}}\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\tilde{\upsilon} - \tilde{\tilde{\upsilon}}\right\|^2_{W_2^1(\Omega)}$$

we obtain the inequality

$$\underline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \ge J(\tilde{\tilde{\upsilon}}, \tilde{\tilde{u}}) + \frac{1}{2}\left\|\tilde{u} - \tilde{\tilde{u}}\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\tilde{\upsilon} - \tilde{\tilde{\upsilon}}\right\|^2_{W_2^1(\Omega)}.$$

From (23) we get the relation $\overline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \le J(\tilde{\upsilon}, \tilde{u})$ which implies $J(\tilde{\tilde{\upsilon}}, \tilde{\tilde{u}}) \le J(\tilde{\upsilon}, \tilde{u})$. This in its turn gives $J(\tilde{\tilde{\upsilon}}, \tilde{\tilde{u}}) = J(\tilde{\upsilon}, \tilde{u})$. The last implies the validity of

$$\frac{1}{2}\left\|\tilde{u} - \tilde{\tilde{u}}\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\tilde{\upsilon} - \tilde{\tilde{\upsilon}}\right\|^2_{W_2^1(\Omega)} = 0 .$$

From this we obtain $\tilde{\tilde{\upsilon}} = \tilde{\upsilon}$, $\tilde{\tilde{u}} = \tilde{u}$.

So,

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \text{ and } \underline{\lim}_{\varepsilon \to 0} J(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon}, \tilde{u}) .$$

From this we obtain

$$J(\tilde{\upsilon}_\varepsilon, \tilde{u}_\varepsilon) \to J(\tilde{\upsilon}, \tilde{u}) . \qquad (29)$$

From this and from the definition of the functional $J(\upsilon, u)$ we come to the validity of (21) and (22). Theorem is proved.

5.Deriving optimality conditions

Theorem 3. Let $\{\tilde{\upsilon}, \tilde{u}\}$ is an optimal pair in problem (1)-(4), (6). Then there exists a triplet $\{\tilde{\upsilon}, \tilde{u}, \psi\}$ that satisfies the following relations

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\upsilon}(x)\frac{\partial \tilde{u}}{\partial x_i}\right) - \tilde{u}^3 = f(x, t),\ (x, t) \in Q ,$$

$$\tilde{u}(x, 0) = u_0(x), \quad \frac{\partial \tilde{u}}{\partial t}(x, 0) = u_1(x), \quad \tilde{u}\big|_\Sigma = 0 ,$$

$$\frac{\partial^2 \psi}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\upsilon}(x)\frac{\partial \psi}{\partial x_i}\right) - 3\tilde{u}^2\psi = (\tilde{u} - u_d)^5 ,$$

$$\psi(x, T) = \frac{\partial \psi}{\partial t}(x,\ T) = 0\ ,\ \Omega,\quad \psi\big|_{\Sigma} = 0\,,$$

$$\tilde{u} \in L_{\infty}(0,\ T; W_2^1(\Omega)),\ \frac{\partial \tilde{u}}{\partial t} \in L_{\infty}(0,\ T; L_2(\Omega)),$$

$$\psi \in L_{\infty}(0,\ T; L_2(\Omega)),\ \frac{\partial \psi}{\partial t} \in L_{\infty}(0,\ T; W_2^{-1}(\Omega)),$$

and integral identity

$$\int_{\Omega}\left[N\tilde{\upsilon} + \int_0^T \sum_{i=1}^n \frac{\partial \psi}{\partial x_i}\frac{\partial \tilde{u}}{\partial x_i}dt\right](\upsilon - \tilde{\upsilon})dx +$$

$$+ N\int_{\Omega}\sum_{i=1}^n \frac{\partial \tilde{\upsilon}}{\partial x_i}\frac{\partial(\upsilon - \tilde{\upsilon})}{\partial x_i}dx \geq 0,\ \ \forall\, \upsilon \in V. \tag{30}$$

Proof. In order to the pair $\{\tilde{\upsilon}_{\varepsilon},\ \tilde{u}_{\varepsilon}\}$ to be an optimal pair in problem (19), (20) is is necessary fulfillment of the following conditions

$$\frac{d}{d\lambda}J_{\varepsilon}^{a}\left(\tilde{\upsilon}_{\varepsilon},\ \tilde{u}_{\varepsilon} + \lambda\xi\right)\Big|_{\lambda=0} = 0,\ \ \forall\, \xi \in C^2(\overline{Q})\,, \tag{31}$$

$$\xi(x,\ 0) = 0\ ,\ \ \frac{\partial \xi(x, 0)}{\partial t} = 0\ ,\ \ \xi\big|_{\Sigma} = 0\,,$$

$$\frac{d}{d\lambda}J_{\varepsilon}^{a}\left(\tilde{\upsilon}_{\varepsilon} + \lambda(\upsilon - \tilde{\upsilon}_{\varepsilon}),\ \tilde{u}_{\varepsilon}\right)\Big|_{\lambda=0} \geq 0\ ,\ \forall\, \upsilon \in V\,,\ \ \tilde{\upsilon}_{\varepsilon} \in V\ . \tag{32}$$

From (31) we get

$$\frac{1}{\varepsilon}\int_{Q}\left(\frac{\partial^2 \tilde{u}_{\varepsilon}}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_{\varepsilon}(x)\frac{\partial \tilde{u}_{\varepsilon}}{\partial x_i}\right) - \tilde{u}_{\varepsilon}^{\ 3} - f(x,\ t)\right)\times$$

$$\times\left(\frac{\partial^2 \xi}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_{\varepsilon}(x)\frac{\partial \xi}{\partial x_i}\right) - 3\xi\tilde{u}_{\varepsilon}^{\ 2}\right)dxdt +$$

$$+\int_{Q}(\tilde{u}_{\varepsilon} - u_d)^5\,\xi\, dxdt + \int_{Q}(\tilde{u}_{\varepsilon} - \tilde{u})\xi dxdt = 0\,. \tag{33}$$

To write this relation in simple form we introduce the function

$$\psi_{\varepsilon} = -\frac{1}{\varepsilon}\left(\frac{\partial^2 \tilde{u}_{\varepsilon}}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_{\varepsilon}(x)\frac{\partial \tilde{u}_{\varepsilon}}{\partial x_i}\right) - \tilde{u}_{\varepsilon}^{\ 3} - f(x,\ t)\right).$$

Then from (33) in Ω for $\xi\big|_{\Sigma}=0$ we obtain the integral identity

$$-\int_Q \psi_\varepsilon\left(\frac{\partial^2\xi}{\partial t^2}-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_\varepsilon(x)\frac{\partial\xi}{\partial x_i}\right)-3\xi\tilde{u}_\varepsilon^{\,2}\right)dxdt+ \\ +\int_Q\left(\tilde{u}_\varepsilon-u_d\right)^5\xi\,dxdt+\int_Q\left(\tilde{u}_\varepsilon-\tilde{u}\right)\xi dxdt=0 \quad , \qquad (34)$$

for $\forall\xi\in C^2(\overline{Q}),\quad \xi(x,\ 0)=0,\quad \dfrac{\partial\xi(x,\,0)}{\partial t}=0$.

And from (32) we get

$$N\int_\Omega\left[\tilde{\upsilon}_\varepsilon(\upsilon-\tilde{\upsilon}_\varepsilon)+\sum_{i=1}^n\frac{\partial\tilde{\upsilon}_\varepsilon}{\partial x_i}\frac{\partial(\upsilon-\tilde{\upsilon}_\varepsilon)}{\partial x_i}\right]dx- \\ -\frac{1}{\varepsilon}\int_Q\left(\frac{\partial^2\tilde{u}_\varepsilon}{\partial t^2}-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_\varepsilon(x)\frac{\partial\tilde{u}_\varepsilon}{\partial x_i}\right)-\tilde{u}_\varepsilon^3-f(x,t)\right)\left(\sum_{i=1}^n\frac{\partial}{\partial x_i}\left((\upsilon-\tilde{\upsilon}_\varepsilon)\frac{\partial\tilde{u}_\varepsilon}{\partial x_i}\right)\right)dxdt+ \\ +\int_\Omega\left[\left(\tilde{\upsilon}_\varepsilon-\tilde{\upsilon}\right)\left(\upsilon-\tilde{\upsilon}_\varepsilon\right)+\sum_{i=1}^n\frac{\partial}{\partial x_i}\left[\left(\tilde{\upsilon}_\varepsilon-\tilde{\upsilon}\right)\frac{\partial}{\partial x_i}\left(\upsilon-\tilde{\upsilon}_\varepsilon\right)\right]\right]dx\ge 0,\ \forall\upsilon\in V. \qquad (35)$$

Considering the definition above (35) can be written in the form

$$\int_\Omega\left[N\tilde{\upsilon}_\varepsilon-\int_0^T\sum_{i=1}^n\frac{\partial\psi_\varepsilon}{\partial x_i}\frac{\partial\tilde{u}_\varepsilon}{\partial x_i}dt\right](\upsilon-\tilde{\upsilon}_\varepsilon)dx+N\int_\Omega\sum_{i=1}^n\frac{\partial\tilde{\upsilon}_\varepsilon}{\partial x_i}\frac{\partial(\upsilon-\tilde{\upsilon}_\varepsilon)}{\partial x_i}dx+ \\ +\int_\Omega\left[\left(\tilde{\upsilon}_\varepsilon-\tilde{\upsilon}\right)\left(\upsilon-\tilde{\upsilon}_\varepsilon\right)+\sum_{i=1}^n\frac{\partial}{\partial x_i}\left[\left(\tilde{\upsilon}_\varepsilon-\tilde{\upsilon}\right)\frac{\partial}{\partial x_i}\left(\upsilon-\tilde{\upsilon}_\varepsilon\right)\right]\right]dx\ge 0\ \ \forall\upsilon\in V. \qquad (36)$$

(34) shows that ψ_ε is a weak solution to the problem

$$\frac{\partial^2\psi_\varepsilon}{\partial t^2}-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\tilde{\upsilon}_\varepsilon\frac{\partial\psi_\varepsilon}{\partial x_i}\right)-3\tilde{u}_\varepsilon^{\,2}\psi_\varepsilon=(\tilde{u}_\varepsilon-u_d)^5+(\tilde{u}_\varepsilon-\tilde{u})\ \text{ in } Q\ , \\ \psi_\varepsilon(x,T)=\frac{\partial\psi_\varepsilon}{\partial t}(x,\ T)=0\ \text{ in } \Omega, \\ \psi_\varepsilon\big|_\Sigma=0. \qquad (37)$$

Basing on the result of [5] one can establish the following estimation for problem (37)

$$\|\psi_\varepsilon\|_{L_\infty(0,T;L_2(\Omega))}+\left\|\frac{\partial\psi_\varepsilon}{\partial t}\right\|_{L_\infty(0,T;W_2^{-1}(\Omega))}\le c.$$

Then if to pass to limit as $\varepsilon \to 0$ in problem (25), (26), inequality (36) and problem (37) considering (21), (22) we obtain the validity of the theorem.
Theorem is proved.

6.Conclusion

The existence of the optimal pair in the problem of optimal control with the leading coefficient of the considered second order hyperbolic equation with a discontinuous solution, the convergence of the adapted penalty method is proved, and a necessary condition for optimality in the form of a variational inequality is derived.

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