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ON CONGRUENCE SCHEMES AND k-MAJORITY ALGEBRAS

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Abstract

Using congruence schemes, we present a characterization of compatible reflexive relations for k-majority algebras (i.e. for algebras having k-ary near unanimity operation among its term operations, $k \geq 3$). For algebras in congruence n-permutable varieties we show that every binary $(n-1)$-pretransitive compatible relation is symmetric and we obtain some consequences for special types of relations.

Keywords: congruence schemes, majority algebra, tolerance lattice

Mathematics Subject Classification *(2020): 08A10, 08A30, 06D15*

Introduction

Congruence scheme is a very important idea in the study of congruence lattices of algebras (see [1] - [3]). A term function $m(x_1, \ldots, x_k)$ of an algebra $\mathbb{A} = (A; F)$ is called k-*majority term* (or k-*ary near unanimity term*; see [4], p.34),

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$k \geq 3$, if

$$m(y, x, x, \dots, x) = m(x, y, x, x, \dots, x) = \cdots = m(x, x, \dots, x, y) = x$$

holds for all $x, y \in A$. Clearly, 3-majority term is exactly the majority term; for example, any lattice $(L;\vee,\wedge)$ admits a majority term. Algebras having a k-majority term are called *k-majority algebras*. In [3], some different characterisations of majority algebras are given. A *quasiorder* of an algebra $\mathbb{A} = (A; F)$ is a reflexive, transitive binary relation $\rho \subseteq A^2$, which is compatible with the operations of $\mathbb{A}$. Let $(\mathrm{Quord}\mathbb{A}; \subseteq)$ stand for the set of quasiorders of $\mathbb{A}$. It is easy to see that $(\mathrm{Quord}\mathbb{A}; \subseteq)$ is an algebraic lattice (here the lattice operation $\wedge$ is the usual set intersection of the binary relations).

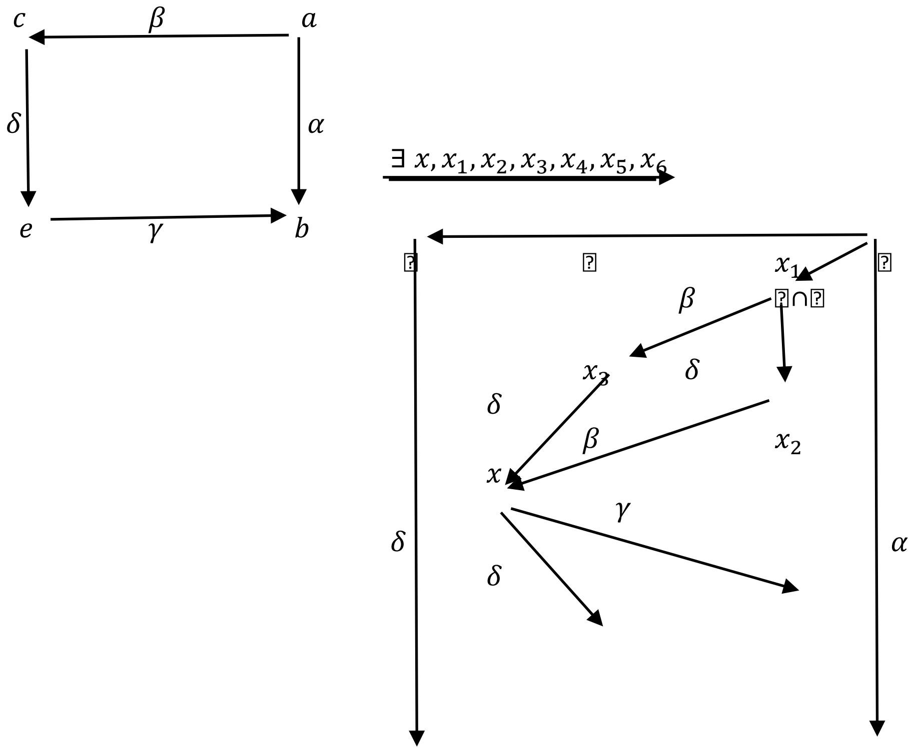
1° 4-majority algebras

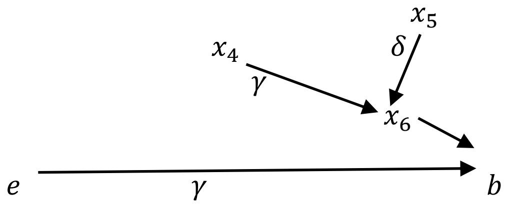
Proposition 1. Let $\mathbb{A} = (A; F)$ be an algebra and consider the following assertions.

(1) $\mathbb{A}$ has a 4-majority term function m.

(2) For every $a, b, c, d \in A$ and any compatible reflexive relations $\alpha, \beta, \delta, \gamma \subseteq A^2$ SCHEME-4 below is satisfied.

SCHEME-4





Here $(a, x_1) \in \alpha \cap \beta$, $(x_1, c) \in \beta$, $(x_6, b) \in \alpha \cap \gamma$.

(3) For any compatible reflexive relations $\alpha, \beta, \gamma, \delta \subseteq A^2$ we have

$$\alpha \cap (\beta \circ \delta \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

(4) For any quasiorders $\alpha, \beta, \delta, \gamma$ of $\mathbb{A}$ we have:

$$\alpha \cap (\beta \circ \delta \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. (1) ⇒ (2). Let $m(t, u, v, w)$ be a 4-majority term function $A^4 \to A$ of algebra $\mathbb{A} = (A; F)$. Take $x = m(a, c, e, b)$, $x_1 = m(a, a, c, b)$, $x_2 = m(a, c, c, b)$, $x_3 = m(a, a, e, b)$, $x_4 = m(a, e, e, b)$, $x_5 = m(a, c, b, b)$ and $x_6 = m(a, e, b, b)$. Then we see that $(a, x_1) = (m(a, a, a, b), m(a, a, c, b)) \in \beta$ and $(a, x_1) = = (m(a, a, c, a), m(a, a, c, b)) \in \alpha$. So, $(a, x_1) \in \alpha \cap \beta$.

Similarly, we find that

$$(x_6, b) = \big(m(a, e, b, b), m(b, e, b, b)\big) = \big(m(a, e, b, b), m(a, b, b, b)\big) \in \alpha \cap \gamma.$$

It is clear also $(x_1, c) = (m(a, a, c, b), m(c, c, c, b)) \in \beta$,

$$(x_1, x_2) = (m(a, a, c, b), m(a, c, c, b)) \in \beta,$$
$$(x_2, c) = (m(a, c, c, b), m(c, c, c, b)) \in \beta,$$
$$(x_1, x_3) = (m(a, a, c, b), m(a, a, e, b)) \in \delta,$$
$$(e, x_4) = (m(a, e, e, e), m(a, e, e, b)) \in \gamma,$$
$$(e, x_6) = (m(a, e, e, e), m(a, e, b, b)) \in \gamma,$$
$$(x_4, x_6) = (m(a, e, e, b), m(a, e, b, b)) \in \gamma,$$
$$(x_5, x_6) = (m(a, c, b, b), m(a, e, b, b)) \in \delta,$$
$$(x_1, b) = (m(a, a, c, b), m(b, b, c, b)) \in \alpha,$$
$$(x_3, b) = (m(a, a, e, b), m(b, b, e, b)) \in \alpha,$$
$$(x_5, b) = (m(a, c, b, b), m(b, c, b, b)) \in \alpha,$$
$$(a, x_3) = (m(a, a, e, a), m(a, a, e, b)) \in \alpha,$$
$$(a, x_5) = (m(a, c, a, a), m(a, c, b, b)) \in \alpha,$$
$$(a, x_6) = (m(a, e, a, a), m(a, e, b, b)) \in \alpha,$$

$$(x_2, x) = (m(a, c, c, b), m(a, c, e, b)) \in \delta,$$
$$(x_3, x) = (m(a, a, e, b), m(a, c, e, b)) \in \beta,$$
$$(x, x_4) = (m(a, c, e, b), m(a, e, e, b)) \in \delta,$$
$$(x, x_5) = (m(a, c, e, b), m(a, c, b, b)) \in \gamma.$$

(2) ⇒ (3). Let $(a, b) \in \alpha \cap (\beta \circ \delta \circ \gamma)$. Then there are elements $c, e \in A$ such that $(a, c) \in \beta$, $(c, e) \in \delta$, $(e, b) \in \gamma$.

Applying SCHEME-4 we obtain elements $x, x_1, \dots, x_6 \in A$ such that:

$$(a, x_1) \in \alpha \cap \beta,\ (x_1, c) \in \beta,\ (x_1, x_2) \in \beta,\ (x_2, c) \in \beta,\ (x_1, x_3) \in \delta,\ (e, x_4) \in \gamma,$$
$$(e, x_6) \in \gamma,\ (x_4, x_6) \in \gamma,\ (x_5, x_6) \in \delta,\ (x_6, b) \in \alpha \cap \gamma.$$

Next,

$(x_1, b) \in \alpha$, $(x_3, b) \in \alpha$, $(x_5, b) \in \alpha$, $(a, x_3) \in \alpha$, $(a, x_5) \in \alpha$, $(a, x_6) \in \alpha$; moreover, $(x_2, x) \in \delta$, $(x_3, x) \in \beta$, $(x, x_4) \in \delta$, $(x, x_5) \in \gamma$.

So, $(a, b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma)$.

Similarly, we get:

$$(a, b) \in (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

Thus

$$\alpha \cap (\beta \circ \delta \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

(3) ⇒ (4). If $\alpha, \beta, \delta, \gamma \in \mathrm{Quord}\mathbb{A}$, then

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq$$
$$\subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq \alpha \circ \alpha \subseteq \alpha$$

and

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq$$
$$\subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq$$
$$\subseteq \beta \circ \beta \circ \delta \circ \delta \circ \gamma \circ \gamma.$$

So, $[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$

$$\cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha] \subseteq \alpha \cap (\beta \circ \delta \circ \gamma).$$

As the converse inclusion holds by assumption, we obtain (4). □

Let $\theta(a, b)$ denotes the principal congruence of an algebra $\mathbb{A} = (A; F)$ generated by the pair $(a, b) \in A \times A$. It is well known that if $\varphi: \mathbb{A} \to \mathbb{B}$ is a homomorphism then $(u, v) \in \theta(a, b)$ implies $(\varphi(u), \varphi(v)) \in \theta(\varphi(a), \varphi(b))$.

Theorem 2. Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent:

(a) $\mathcal{V}$ has a 4-majority term.

(b) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies SCHEME-4.

(c) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any compatible reflexive relations $\alpha, \beta, \delta, \gamma \subseteq A^2$ we have

$$\alpha \cap (\beta \circ \delta \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

(d) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and every $\alpha, \beta, \delta, \gamma \in Con\mathbb{A}$ satisfy the equality

$$\alpha \cap (\beta \circ \delta \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta)) \cap \alpha] \circ (\alpha \cap \gamma) \cap \cap (\alpha \cap \beta) \circ [((\beta \circ \delta \cap \delta \circ \beta) \circ (\delta \circ \gamma \cap \gamma \circ \delta) \circ (\alpha \cap \gamma)) \cap \alpha].$$

Proof By Proposition 1, (a) implies (b), (b) implies (c) and (c) implies (d), as $Con\mathbb{A} \subseteq Quord\mathbb{A}$.

(d) implies (a). Now consider the free algebra $\mathbb{F}_{\mathcal{V}}(x, y, z, t) \in \mathcal{V}$. As

$$(x, t) \in \theta(x, t) \cap (\theta(x, y) \circ \theta(y, z) \circ \theta(z, t)),$$

the assumption (d) implies:

$$(x, t) \in [(\theta(x, t) \cap \theta(x, y)) \circ ((\theta(x, y) \circ \theta(y, z) \cap \theta(y, z) \circ \theta(x, y)) \circ \circ \big(\theta(y, z) \circ \theta(z, t) \cap \theta(z, t) \circ \theta(y, z)\big) \cap (\theta(x, t)] \circ \circ (\theta(x, t) \cap \theta(z, t)) \cap (\theta(x, t) \cap (\theta(x, y)) \circ [((\theta(x, y) \circ \theta(y, z) \cap \theta(y, z) \circ \theta(x, y)) \circ \circ (\theta(y, z) \circ \theta(z, t) \cap \theta(z, t) \circ \theta(y, z)) \circ (\theta(x, t) \cap \theta(z, t))) \cap \theta(x, t)].$$

Hence there is a term $m(x, y, z, t) \in \mathbb{F}_{\mathcal{V}}(x, y, z, t)$ such that

$$x(\theta(x, t) \cap \theta(x, y))m(x, x, y, t),$$
$$m(x, x, y, t)\theta(x, y)m(x, y, y, t)\theta(y, z)m(x, y, z, t) \quad \& \quad \& \quad m(x, x, y, t)\theta(y, z)m(x, x, z, t)\theta(x, y)m(x, y, z, t)$$

and

$$m(x, y, z, t)\, \theta(y, z)m(x, z, z, t)\theta(z, t)m(x, z, t, t) \quad \& \quad \& \quad m(x, y, z, t)\, \theta(z, t)\, m(x, y, t, t)\theta(y, z)\, m(x, z, t, t);$$

next $m(x, z, t, t)\big(\theta(x, t) \cap \theta(z, t)\big)t$;

moreover, $m(x, x, y, t)\theta(x, t)t \quad \& \quad x\theta(x, t)m(x, z, t, t)$.

Now, using the endomorphism $\varphi : \mathbb{F}_{\mathcal{V}}(x, y, z, t) \to \mathbb{F}_{\mathcal{V}}(x, y, z, t)$ with $\varphi(x) = \varphi(y) = x$, $\varphi(z) = z$, $\varphi(t) = t$ from $(x, m(x, x, y, t)) \in \theta(x, y)$ we obtain:

$$\big(x, m(x,x,x,t)\big) = (\varphi(x), m(\varphi(x), \varphi(x), \varphi(y), \varphi(t))) =$$
$$= \Big(\varphi(x), \varphi\big(m(x,x,y,t)\big)\Big) \in \theta\big(\varphi(x), \varphi(y)\big) = \theta(x,x) = \Delta.$$

Thus $x = m(x,x,x,t)$. The identities

$$x = m(x,x,y,x) = m(x,y,x,x) = m(y,x,x,x)$$

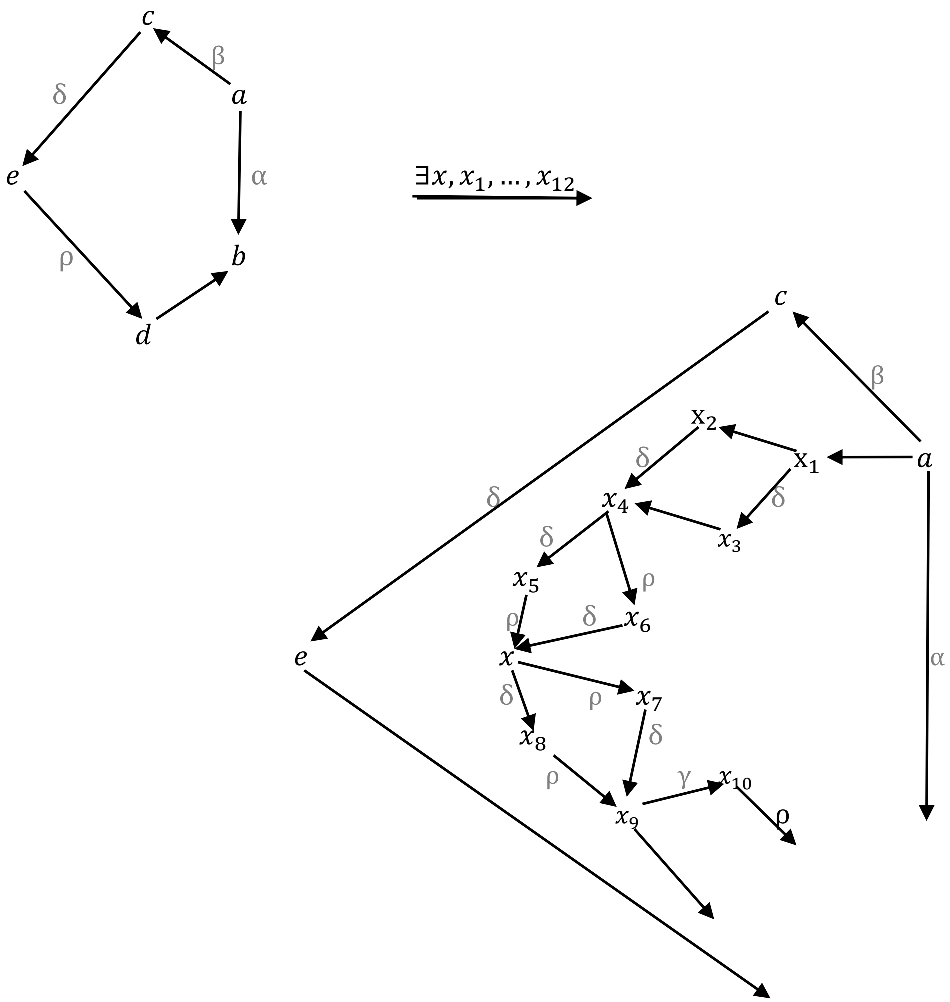
can be proved in a similar way. □

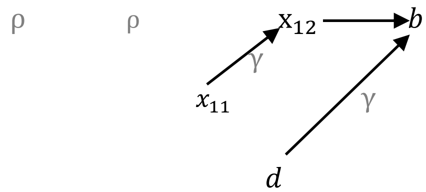
2° 5-majority algebras

Proposition 3. Let $\mathbb{A} = (A; F)$ be an algebra and consider the following assertions.

(5) $\mathbb{A}$ has a 5-majority term function m.

(6) For every $a, b, c, e, d \in A$ and any compatible reflexive relations $\alpha, \beta, \delta, \rho, \gamma \subseteq A^2$ SCHEME-5 below is satisfied.





Here $(a, x_1) \in \alpha \cap \beta$, $(x_1, x_2) \in \beta$, $(x_3, x_4) \in \beta$, $(x_{11}, x_{12}) \in \gamma$, $(x_{12}, b) \in \alpha \cap \gamma$.

(7) All compatible reflexive binary relations $\alpha, \beta, \delta, \rho, \gamma \subseteq A^2$ satisfy

$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ \ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$(\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap (\ \alpha \cap \beta) \circ [\alpha \cap ((\ \beta \circ \delta \cap \delta \circ \beta) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

(8) For every quasiorders $\alpha, \beta, \delta, \rho, \gamma$ of algebra $\mathbb{A}$ we have:

$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ \ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ$$
$$\circ (\alpha \cap \gamma) \cap (\ \alpha \cap \beta) \circ [\alpha \cap ((\ \beta \circ \delta \cap \delta \circ \beta) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Then (5) ⇒ (6) ⇒ (7) ⇒ (8).

Proof. (5) ⇒ (6). Let $m(s, t, u, v, w): A^5 \to A$ be a 5-majority term function of algebra $\mathbb{A}$. Suppose $(a, b) \in \alpha$, $(a, c) \in \beta$, $(c, e) \in \delta$, $(e, d) \in \rho$ and $(d, b) \in \gamma$ for the elements $a, b, c, e, d \in A$ and let $\alpha, \beta, \delta, \rho, \gamma \subseteq A^2$ are compatible reflexive relations of $\mathbb{A}$. Take $x = m(a, c, e, d, b)$, $x_1 = m(a, a, a, c, b)$, $x_2 = m(a, c, c, c, b)$,

$$x_3 = m(a, a, a, e, b),\ x_4 = m(a, c, c, e, b),\ x_5 = m(a, c, e, e, b),$$
$$x_6 = m(a, c, c, d, b),\ x_7 = m(a, c, d, d, b),\ x_8 = m(a, e, e, d, b),$$
$$x_9 = m(a, e, d, d, b),\ x_{10} = m(a, e, b, b, b),\ x_{11} = m(a, d, d, d, b).$$
$$x_{12} = m(a, d, b, b, b).$$

Then we obtain:

$$(a, x_1) = \big(m(a, a, a. a, b), m(a, a, a, c, b)\big) =$$
$$= (m(a, a, a. c, a), m(a, a, a, c, b)) \in \alpha \cap \beta.$$
$$(x_{12}, b) = \big(m(a, d, b, b, b), m(a, b, b, b, b)\big) =$$
$$= \big(m(a, d, b, b, b), m(b, d, b, b, b)\big) \in \alpha \cap \gamma.$$

Also,

$$(x_1, c) = \big(m(a, a, a, c, b), m(c, c, c, c, b)\big) \in \beta,$$
$$(x_2, c) = \big(m(a, c, c, c, b), m(c, c, c, c, b)\big) \in \beta,$$
$$(x_1, x_2) = \big(m(a, a, a, c, b), m(a, c, c, c, b)\big) \in \beta,$$
$$(x_2, x_4) = \big(m(a, c, c, c, b), m(a, c, c, e, b)\big) \in \delta,$$

$$(x_1, x_3) = \big(m(a,a,a,c,b), m(a,a,a,e,b)\big) \in \delta,$$
$$(x_3, x_4) = \big(m(a,a,a,e,b), m(a,c,c,e,b)\big) \in \beta,$$
$$(x_4, x_5) = \big(m(a,c,c,e,b), m(a,c,e,e,b)\big) \in \delta,$$
$$(x_4, x_6) = \big(m(a,c,c,e,b), m(a,c,c,d,b)\big) \in \rho,$$
$$(x_5, x) = \big(m(a,c,e,e,b), m(a,c,e,d,b)\big) \in \rho,$$
$$(x_6, x) = \big(m(a,c,c,d,b), m(a,c,e,d,b)\big) \in \delta,$$
$$(x, x_7) = \big(m(a,c,e,d,b), m(a,c,d,d,b)\big) \in \rho,$$
$$(x, x_8) = \big(m(a,c,e,d,b), m(a,e,e,d,b)\big) \in \delta,$$
$$(x_8, x_9) = \big(m(a,e,e,d,b), m(a,e,d,d,b)\big) \in \rho,$$
$$(x_7, x_9) = \big(m(a,c,d,d,b), m(a,e,d,d,b)\big) \in \delta,$$
$$(x_9, x_{10}) = \big(m(a,e,d,d,b), m(a,e,b,b,b)\big) \in \gamma,$$
$$(x_9, x_{11}) = \big(m(a,e,d,d,b), m(a,d,d,d,b)\big) \in \delta,$$
$$(x_{10}, x_{12}) = \big(m(a,e,b,b,b), m(a,d,b,b,b)\big) \in \rho,$$
$$(x_{11}, x_{12}) = \big(m(a,d,d,d,b), m(a,d,b,b,b)\big) \in \gamma,$$
$$(a, x_{12}) = \big(m(a,d,a,a,a), m(a,d,b,b,b)\big) \in \alpha,$$
$$(x_1, b) = \big(m(a,a,a,c,b), m(b,b,b,c,b)\big) \in \alpha,$$
$$(a, x_{10}) = \big(m(a,e,a,a,a), m(a,e,b,b,b)\big) \in \alpha,$$
$$(x_2, b) = \big(m(a,a,a,e,b), m(b,b,b,e,b)\big) \in \alpha.$$

(6) ⇒ (7). Let $(a,b) \in \alpha \cap (\beta \circ \delta \circ \rho \circ \gamma)$. Then there are elements $c, d, e \in A$ such that $(a,c) \in \beta$, $(c,e) \in \delta$, $(e,d) \in \rho$ and $(d,b) \in \gamma$. Also, $(a,b) \in \alpha$. By applying SCHEME-5 we obtain elements $x, x_1, \ldots, x_{12} \in A$ such that $(a, x_1) \in \alpha \cap \beta$, $(x_1, x_2) \in \beta$, $(x_1, x_3) \in \delta$, $(x_2, x_4) \in \delta$, $(x_3, x_4) \in \beta$, $(x_4, x_5) \in \delta$, $(x_4, x_6) \in \rho$, $(x_5, x) \in \rho$, $(x_6, x) \in \delta$, $(x, x_7) \in \rho$, $(x, x_8) \in \delta$, $(x_8, x_9) \in \rho$, $(x_7, x_9) \in \delta$, $(x_9, x_{10}) \in \gamma$, $(x_9, x_{11}) \in \delta$, $(x_{10}, x_{12}) \in \rho$, $(x_{11}, x_{12}) \in \gamma$, $(x_{12}, b) \in \alpha \cap \gamma$, $(x_1, b) \in \alpha$, $(a, x_{12}) \in \alpha$.

So,

$$(a,b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma).$$

Similarly, we get:

$$(a,b) \in (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$

$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Thus,

$$(a, b) \in [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Hence, the inclusion in (7) is proved.

(7) ⇒ (8). If $\alpha, \beta, \delta, \rho, \gamma \in Quard(\mathbb{A})$ then

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq$$
$$\subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \subseteq \alpha \circ \alpha \subseteq \alpha$$

and

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq$$
$$(\alpha \cap \beta) \circ \beta \circ \delta \circ \delta \circ \rho \circ \rho \circ \gamma \circ \gamma \subseteq \beta \circ \beta \circ \delta \circ \rho \circ \gamma \subseteq \beta \circ \delta \circ \rho \circ \gamma.$$

Thus

$$[((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))] \subseteq \alpha \cap (\beta \circ \delta \circ \rho \circ \gamma).$$

As the converse inclusion holds by the assumption, we obtain (8). □

Theorem 4. Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent:

(a) $\mathcal{V}$ has a 5-majority term.

(b) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies SCHEME-5.

(c) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and any compatible reflexive relations $\alpha, \beta, \delta, \rho, \gamma \subseteq A^2$ we have

$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) \subseteq [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ$$

$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

(d) For any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ and every $\alpha, \beta, \delta, \rho, \gamma \in Con\mathbb{A}$ satisfy the equality:

$$\alpha \cap (\beta \circ \delta \circ \rho \circ \gamma) = [((\alpha \cap \beta) \circ (\beta \circ \delta \cap \delta \circ \beta) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma)) \cap \alpha] \circ (\alpha \cap \gamma) \cap$$
$$\cap (\alpha \cap \beta) \circ [\alpha \cap ((\beta \circ \delta \cap \delta \circ \beta) \circ$$
$$\circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\rho \circ \delta \cap \delta \circ \rho) \circ (\gamma \circ \rho \cap \rho \circ \gamma) \circ (\alpha \cap \gamma))].$$

Proof. In view of Proposition 3, (a) implies (b) and (b) implies (c). Proposition 3 also gets that (c) implies (d), as $Con\mathbb{A} \subseteq Quard\mathbb{A}$.

(d) implies (a). Consider now the free algebra $\mathbb{F}_{\mathcal{V}}(x, y, z, u, t) \in \mathcal{V}$. As

$$(x, t) \in \theta(x, t) \cap (\theta(x, y) \circ \theta(y, z) \circ \theta(z, u) \circ \theta(u, t)),$$

the assumption of (d) implies:

$$(x, t) \in[(\theta(x, t) \cap (\theta(x, y)) \circ (((\theta(x, y) \circ \theta(y, z) \cap \theta(y, z) \circ \theta(x, y)) \circ$$
$$\circ (\theta(y, z) \circ \theta(z, u) \cap \theta(z, u) \circ \theta(y, z)) \circ (\theta(z, u) \circ \theta(u, t) \cap \theta(u, t) \circ \theta(z, u)))$$
$$\cap \theta(x, t)] \circ$$
$$\circ (\theta(x, t) \cap \theta(u, t)) \cap (\theta(x, t) \cap \theta(x, y)) \circ [((\theta(x, y) \circ \theta(y, z) \cap \theta(y, z)$$
$$\circ \theta(x, y)) \circ$$
$$\circ (\theta(y, z) \circ \theta(z, u) \cap \theta(z, u) \circ \theta(y, z)) \circ (\theta(z, u) \circ \theta(u, t) \cap \theta(u, t) \circ \theta(z, u)) \circ$$
$$\circ (\theta(x, t) \cap \theta(u, t))) \cap \theta(x, t)].$$

Hence there is a term $m(x, y, z, u, t) \in \mathbb{F}_{\mathcal{V}}(x, y, z, u, t)$ such that:

$$x\big(\theta(x, t) \cap \theta(x, y)\big)m(x, x, x, y, t)\theta(x, y)m(x, y, y, y, t)\theta(y, z)m(x, y, y, z, t) \;\&$$
$$\& \; m(x, x, x, y, t)\theta(y, z)m(x, x, x, z, t)\theta(x, y)m(x, y, y, z, t)$$

and

$$m(x, y, y, z, t)\theta(y, z)m(x, y, z, z, t)\theta(z, u)m(x, y, z, u, t) \;\&$$
$$\& \; m(x, y, y, z, t) \, \theta(z, u)m(x, y, y, u, t) \, \theta(y, z)m(x, y, z, u, t).$$

Next,

$$m(x, y, z, u, t)\theta(y, z)m(x, z, z, u, t)\theta(z, u)m(x, z, u, u, t)\theta(z, u)$$
$$\theta(z, u)m(x, u, u, u, t)\theta(u, t)\theta(u, t)m(z, u, t, t, t) \;\;\&$$
$$\&m(x, y, z, u, t)\theta(z, u)m(x, y, u, u, t)\theta(y, z)m(x, z, u, u, t)\theta(u, t)$$

$\theta(u, t)m(x, z, t, t, t)\theta(z, u)m(x, u, t, t, t)$, and $m(x, u, t, t, t)\big(\theta(x, t) \cap \theta(u, t)\big)t$.

Now, using the endomorphism φ: $\mathbb{F}_{\mathcal{V}}(x, y, z, u, t) \to \mathbb{F}_{\mathcal{V}}(x, y, z, u, t)$ with

$\varphi(x) = \varphi(y) = x$, $\varphi(z) = z$, $\varphi(u) = u$, $\varphi(t) = t$ and from $(x, m(x, x, x, y, t)) \in \theta(x, y)$ we obtain:

$$\big(x, m(x, x, x, x, t)\big) = (\varphi(x), m(\varphi(x), \varphi(x), \varphi(x), \varphi(y), \varphi(t))) =$$

$$= \Big(\varphi(x), \varphi\big(m(x, x, x, y, t)\big)\Big) \in \theta\big(\varphi(x), \varphi(y)\big) = \theta(x, x) = \Delta.$$

Thus, $x = m(x, x, x, x, t)$. The identities $x = m(x, x, x, y, x) = m(x, x, y, x, x) =$

$$= m(x, y, x, x, x) = m(y, x, x, x, x)$$

can be proved in a similar way. □

Remark. Also, it is easy to see that under conditions of Proposition 3 ((5) implies (6)) we have the following relations:

$$x_2 \delta x_{13},\ x_4 \delta x_{13},\ x_5 \delta x_{13} \quad \text{and} \quad x_{13} \rho x_8,\ x_{13} \rho x_9,\ x_{13} \rho x_{11};$$

here $x_{13} = m(a, e, e, e, b)$.

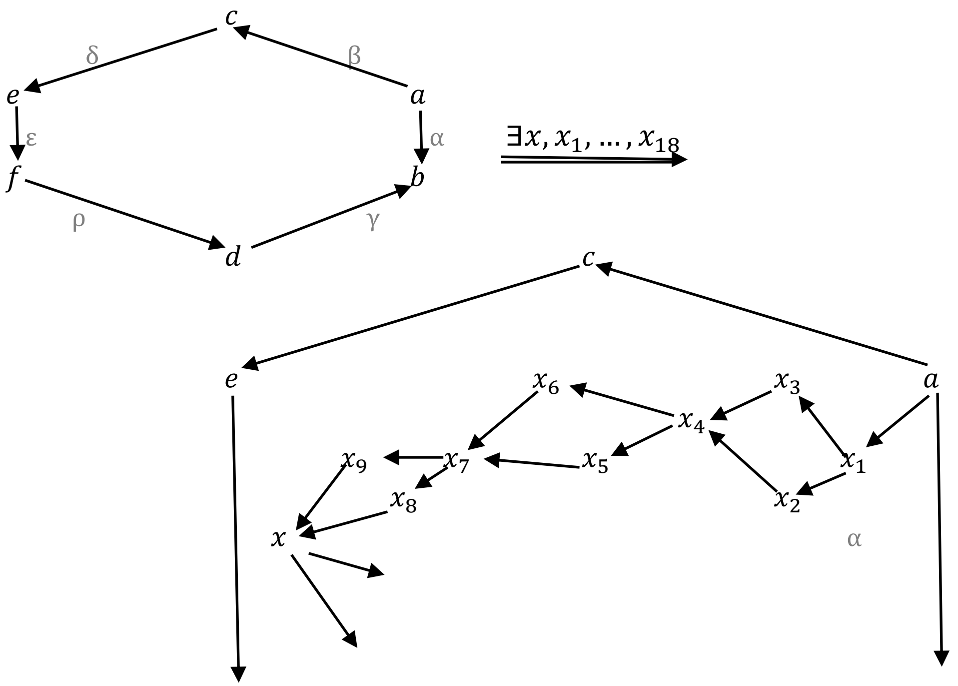
3° On 6-majority algebras

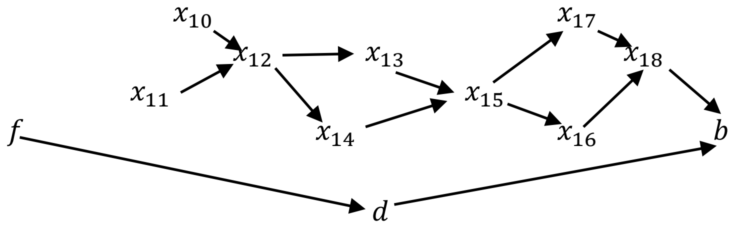
Using the methods given in the proofs of Theorems 2 and 4, one can prove a similar theorem for varieties of algebras with 6-majority terms (we show only first two parts):

Theorem 5. Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent:

(a) $\mathcal{V}$ has a 6-majority term.

(b) Any algebra $\mathbb{A} = (A; F) \in \mathcal{V}$ satisfies the following SCHEME-6:





Here $(a, x_1) \in \alpha \cap \beta$, $(x_1, x_3) \in \beta$, $(x_1, x_2) \in \delta$, $(x_2, x_4) \in \beta$, $(x_3, x_4) \in \delta$, $(x_4, x_6) \in \delta$, $(x_4, x_5) \in \varepsilon$, $(x_6, x_7) \in \varepsilon$, $(x_5, x_7) \in \delta$, $(x_7, x_9) \in \varepsilon$, $(x_7, x_8) \in \rho$, $(x_9, x) \in \rho$, $(x_8, x) \in \varepsilon$, $(x, x_{11}) \in \varepsilon$, , $(x, x_{10}) \in \delta$, $(x_{10}, x_{12}) \in \varepsilon$, $(x_{11}, x_{12}) \in \delta$, $(x_{12}, x_{13}) \in \varepsilon$, $(x_{12}, x_{14}) \in \rho$, $(x_{14}, x_{15}) \in \varepsilon$, $(x_{13}, x_{15}) \in \rho$, $(x_{15}, x_{16}) \in \rho$, $(x_{15}, x_{17}) \in \gamma$, $(x_{17}, x_{18}) \in \rho$, $(x_{16}, x_{18}) \in \gamma$, $(x_{18}, b) \in \alpha \cap \gamma$. □

Moreover, it can be shown that there is an algorithm for constructing congruence schemes in the case of a variety of algebras with a k-majority term for an arbitrary k. In this case, the number of internal “rhombs” is equal to 2(k-3).

4° On compatible relations

Definition. Let θ be any binary relation which is compatible with all operations of an algebra $\mathbb{A} = (A; F)$. If the relation $\theta^n = \theta \circ \theta \circ \dots \circ \theta$ (n times) is contained in θ, $\theta^n \subseteq \theta$, then we will say that θ is *n-pretransitive relation* of $\mathbb{A}$. Of course, every compatible relation is 1-pretransitive.

Theorem 6. If $\mathbb{A}$ is an algebra in a n-permutable variety then every $(n-1)$-pretransitive relation of $\mathbb{A}$ is a symmetric relation.

Proof. Let $(a, b) \in \theta$ and θ is a $(n-1)$-pretransitive relation of $\mathbb{A}$. Assume that there exist (θ-preserving) terms $p_1, \dots, p_{n-1}$ of $\mathbb{A}$ satisfying the identities:

$$(*) \qquad \begin{cases} x = p_1(x, z, z), \ \ p_{n-1}(x, x, z) = z, \\ \ \ p_i(x, x, z) = p_{i+1}(x, z, z) \ \ \forall i. \end{cases}$$

for n-permutability (a result of Hagemann and Mitshke). We must show that $(b, a) \in \theta$. Indeed by $(*)$ we have:

$$b = p_{n-1}(a, a, b)\theta p_{n-1}(a, b, b) = p_{n-2}(a, a, b)\theta p_{n-2}(a, b, b) = \cdots$$
$$\dots = p_2(a, a, b)\theta p_2(a, b, b) = p_1(a, a, b)\theta p_1(a, b, b) = a.$$

As $\theta^n \subseteq \theta$, we have $(b,a) \in \theta$, as required. □

Definition. Let θ be any binary relation on a set A. If for a pair, $(a,b) \in A \times A$ there is an element $c \in A$ such that $(a,c) \in \theta$ and $(b,c) \in \theta$ then we will say that (a,b) *has a right θ-bound*.

Lemma 7. Let a binary relation θ is compatible with the operations of an algebra $\mathbb{A}$ and let $p: A^3 \to A$ be a term of $\mathbb{A}$ satisfying the identity $p(x,x,y) = y$. If $(a,b) \in A \times A$ has a right θ-bound, then for all $z \in A$ $p(a,b,z)\theta z$.

The proof is obvious: indeed, let c be a right θ-bound for (a,b) in $\mathbb{A}$. Then for all $z \in A$ we have: $p(a,b,z)\theta p(c,c,z) = z$.

Corollary 8. Let a binary relation θ is compatible with the operations of an algebra $\mathbb{A}$ and let $p: A^3 \to A$ be a term of $\mathbb{A}$ satisfying the identity $p(x,x,y) = y$. If $(a,b) \in A \times A$ has a right θ-bound, then for all $z \in A$ $p(a,b,z)\theta z$

Now we immediately obtain the next result, but firstly we recall the Maltsev condition given by H.-P.Gumm: a variety $\mathcal{V}$ is congruence modular if and only if for some $n \geq 1$ there exist 3-ary terms $d_0, \dots, d_n$ and p such that $\mathcal{V}$ satisfies

(G1) $d_0(x,y,z) = x$,
(G2) $d_i(x,y,x) = x$ for all i,
(G3) $d_i(x,x,y) = d_{i+1}(x,x,y)$ for i even,
(G4) $d_i(x,y,y) = d_{i+1}(x,y,y)$ for i odd,
(G5) $d_n(x,y,y) = p(x,y,y)$,
(G6) $d_0(x,x,y) = y$.

Proposition 9. Let a binary relation θ is compatible with the operations of an algebra $\mathbb{A}$ and assume that every pair $(a,b) \in A \times A$ has a right θ-bound. If the variety $var(\mathbb{A})$ generated by $\mathbb{A}$ is congruence modular then $\mathbb{A}$ satisfies the equations (G1)-(G4) and the relation

$$(\text{G5}')\ \forall x,y\ d_n(x,y,y)\theta y.$$

Note. In a special case when θ is a compatible ordering relation, congruence modularity implies congruence distributivity (see [5]).

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