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GLOBAL BIFURCATION OF SOLUTIONS OF SOME FOURTH-ORDER NONLINEAR EIGENVALUE PROBLEMS WITH SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

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Abstract

We consider nonlinear eigenvalue problems for ordinary differential equations of fourth order with a spectral parameter in the boundary conditions. The global bifurcation from zero of nontrivial solutions to these problems is studied. The existence of two families of unbounded continua of solutions branching from points of a line of trivial solutions and contained in classes of functions with fixed oscillation count is established.

Keywords: nonlinear eigenvalue problem, global bifurcation, unbounded continua, fixed oscillation count.

Mathematics Subject Classification (2020): 34B05, 34B08, 34B09, 34L10, 34L15, 47A75, 47B50, 74H45

1. Introduction

In this paper we consider the following nonlinear eigenvalue problem

$$\ell(y) \equiv y^{(4)} - (q(x)y')' = \lambda y + h(x, y, y', y'', y''', \lambda), \;\; x \in (0, 1), \tag{1}$$

$$y''(0) = y''(1) = 0, \tag{2}$$

$$Ty(0) - a\lambda y(0) = 0, \; Ty(1) - c\lambda y(0) = 0, \tag{3}$$

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where $\lambda \in R$ is an eigen value parameter, $Ty \equiv y''' - qy'$, $q(x)$ is a positive and absolutely continuous function on $[0, 1]$, a, c are real constants such that $a > 0$ and $c < 0$. We suppose that the nonlinear term has the form $h = f + g$, where f and g are real-valued continuous functions on $[0, 1] \times R^5$, satisfying the following conditions: there exists positive constant M and small positive constant τ such that

$$\frac{|f(x, u, s, \vartheta, w, \lambda)|}{|u|} \leq M, \; x \in [0, 1], (u, s, \vartheta, w) \in R^4, u \neq 0, \\ |u| + |s| + |\vartheta| + |w| \leq \tau, \; \lambda \in R; \tag{4}$$

for every bounded interval $\Lambda \subset R$,

$$g(x, u, s, \vartheta, w, \lambda) = o(|u| + |s| + |\vartheta| + |w|), \quad \text{as } |u| + |s| + |\vartheta| + |w| \to 0, \tag{5}$$

uniformly for $(x, \lambda) \in [0,1] \times R$.

In modern mathematics, the theory of bifurcation of nonlinear eigenvalue problems for ordinary differential equations plays an important role, since such problems arise in the study of various processes in mechanics and physics. Note that recently significant results have been obtained in this direction, which were used in problems of the theory of vibrations, thermal convection, hydrodynamics, the theory of critical operating modes of nuclear and chemical reactors, critical loads, etc. (see, for example, [2, 10, 12-14] and references therein). Note that the nonlinear problem (1)-(3) he bifurcation problem (1)–(2) arises when studying the loss of stability of an inhomogeneous rod in the sections of which a longitudinal force acts, and at the ends there are tracking forces (see [10, 14]).

The global bifurcation of solutions to nonlinear eigenvalue problems for second-order ordinary differential equations is studied in detail in [8, 18-20]. This was served by Rabinowitz's alternative and Rabinowitz's construction of classes of functions that have the usual Sturm nodal properties. Recently, similar classes of functions were constructed by Aliyev [2], with the help of which the global bifurcation of solutions to nonlinear eigenvalue problems for ordinary differential equations of fourth order was studied in detail (see also [15, 17]). Note that the corresponding results in the case when the spectral parameter is contained in one boundary condition were obtained in [1, 3, 5, 9]. But in the case when the spectral parameter is contained in two boundary conditions, the global bifurcation of solutions to nonlinear eigenvalue problems for ordinary differential equations of the second and fourth orders can be said to have not been studied.

The main purpose of this paper is to study the global bifurcation of solutions from zero to the nonlinear problem (1)-(3).

This paper is organized as follows. Section 2 gives some some necessary facts and statements. In addition, using Prüfer-type angular functions, we construct classes of functions $S_k^\nu, k \in \mathrm{N}, k \geq 2,$ of $C^3[0, 1]]$ that have oscillatory properties of eigenfunctions of the linear problem obtained from (1)-(3) by setting $g \equiv 0$ and their derivatives. In Section 3, using the approximation technique and the results of papers [11], [18] and [19], the existence of two families of unbounded continua of the set of solutions to problem (1)–(3) contained in these classes is proved.

2. Preliminary results

We consider the following linear spectral problem

$$\begin{gathered} \ell(y) \equiv y^{(4)} - (q(x)y')' = \lambda y \;\; x \in (0, 1), \\ y''(0) = y''(1) = 0, Ty(0) - a\lambda y(0) = 0, Ty(1) - c\lambda y(0) = 0. \end{gathered} \tag{6}$$

Spectral problem (6) was studied in detail in [6], where, in particular, it was shown that the eigenvalues of this problem are real and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$ such that

$$0 = \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots .$$

Moreover, for each $k \in \mathrm{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k has $k-1$ simple zeros in the interval $(0, 1)$.

Let $E = C^3[0, l] \cap BC^0$ be the Banach space equipped with the usual norm $\|u\|_3 = \sum_{i=0}^{3} \|u^{(s)}\|_0$, where $\|u\|_0 = \max_{x \in [0,1]} |u(x)|$ and BC^0 is a set of functions that satisfy boundary conditions (2).

Following [2] let S be the subset of E given by

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(x) \neq 0,\ Tu(x) \neq 0,\ x \in [0, 1],\ i = 0,1,2\},$$

$S_2 = \{u \in E : \exists i_0 \in \{0,1,2\}, x_0 \in (0, 1)$ such that $u^{(i_0)}(x_0) = 0 \vee Tu(x_0) = 0,$ and if $u'(x_0)Tu(x_0) = 0,$ then $u(x)u''(x) < 0$ in a neighbourhood of x_0, and if $u(x_0)u''(x_0) = 0,$ then $u'(x)Tu(x) < 0$ in a neighbourhood of $x_0\}$.

As in [7], we consider the following Prüfer-type transformation

$$\begin{cases} u(x) = \rho(x)\sin\psi(x)\cos\theta(x), \\ u'(x) = \rho(x)\cos\psi(x)\sin\varphi(x), \\ u''(x) = \rho(x)\cos\psi(x)\cos\varphi(x), \\ Tu(x) = \rho(x)\sin\psi(x)\sin\theta(x). \end{cases} \tag{7}$$

Direct calculations show that If $u \in S,$ then the Jacobian

$$J(u) = \rho^3(x)\sin\psi(x)\cos\psi(x)$$

of the transformation (7) does not vanish in $(0, 1)$.

For any $u \in S$ we define the continuous functions $\rho(u,x),\ \theta(u,x),\ \varphi(u,x)$ and $w(u,x),\ x \in [0,\ l],$ as follows [2]:

$$\rho(u,x) = u^2(x) + u'^2(x) + u''^2(x) + (Tu(x))^2,$$

$$\theta(u,x) = \arctan Tu(x)/u(x),$$

$$\varphi(u,x) = \arctan u'(x)/u''(x),\ \varphi(u,0) = \pi/2,$$

$$w(u,x) = \cot\,\psi(u,x) = \frac{u'(x)\cos\theta(u,x)}{u(x)\sin\varphi(u,x)},$$

where we take

$$\psi(u,x) \in (0,\ \pi/2) \text{ for } x \in (0, 1) \text{ when } u(0)u'(0) > 0,$$

$$\psi(u,x) \in (\pi/2,\ \pi) \text{ for } x \in (0, 1) \text{ when } u(0)u'(0) < 0.$$

Let $\alpha(\lambda) = a\lambda$ and $\beta(\lambda) = c\lambda$. Since $a > 0$ and $c < 0$, the function $\alpha(\lambda)$ is strictly increasing, and the function $\beta(\lambda)$ is strictly decreasing on R.

We define continuous functions $\sigma(\lambda)$ and $\omega(\lambda), \lambda \in R,$ by

$$\sigma(\lambda) = \arctan\alpha(\lambda) \text{ and } \omega(\lambda) = \arctan\beta(\lambda),$$

respectively. By the above argument the function $\sigma(\lambda)$ is is strictly increasing, and $\omega(\lambda)$ is strictly decreasing on R. Moreover, the following relations hold:

$$\lim_{\lambda\to\pm\infty}\sigma(\lambda) = \pm\pi/2, \qquad \lim_{\lambda\to\pm\infty}\omega(\lambda) = \mp\pi/2.$$

For each $k \in \mathrm{N},\ k \geq 2,$ each $\nu \in \{+,-\}$ and each $\lambda \in R$ by $S^{\nu}_{k,\lambda}$ we denote the set of functions $u \in S$ that satisfy the following conditions:

(i) $\theta(u,\ 0) = \sigma(\lambda);$ (ii) $\theta(u,\ 1) = \omega(\lambda) + k\pi;$ (iii) $\varphi(u,0) = \pi/2;$

(iv) $\varphi(u,1) = k\pi + \pi/2$ or $\varphi(u,1) = k\pi - \pi/2$ if $\psi\,(u,0) \in [0,\ \pi/2),$ and $\varphi(u,1) = k\pi - \pi/2$ or $\varphi(u,1) = k\pi - 3\pi/2$ if $\psi\,(u,) \in [\,\pi/2, \pi).$

(v) for fixed u, as x increases from 0 to 1, the function $\theta(u,x)$ (respectively, $\varphi(u,x)$) strictly increasing takes values of $m\pi/2, m \in \mathrm{Z}$ (respectively, $s\pi, s \in \mathrm{Z}$); as x decreases, the function $\theta(u,x)$ (respectively, $\varphi(u,x)$) strictly decreasing takes values of $m\pi/2, m \in \mathrm{Z}$ (respectively, $s\pi, s \in \mathrm{Z}$).

(vi) the function $\nu\, u(x)$ is positive in a deleted neighbourhood of $x = 0.$

Now for each $k \in \mathrm{N},\ k \geq 2,$ and each $\nu \in \{+,-\}$ we denote the set S^{ν}_{k} as follows:

$$S^{\nu}_{k} = \bigcup_{\lambda \in R} S^{\nu}_{k,\lambda}.$$

It is obvious that the sets $S^{\nu}_{k}, k \in \mathrm{N}, k \geq 2,$ are pairwise disjoint open subsets of E. Moreover, it follows from [1, Lemma 2.2] that if $u \in \partial S^{\nu}_{k}, k \in \mathrm{N}, k \geq 2,$ then the function u has at least one zero in the interval $(0,\ 1)$ of multiplicity four.

Let

$$\hat{E} = E \oplus R^2 = \{\{u,m,n\}: u \in E,\ m \in R, n \in R\}$$

be the Banach space with the norm $\|\hat{u}\| = \|u\|_3 + |m| + |n|$. We denote by $\hat{S}$ the subset of $\hat{E}$ defined as follows:

$$\hat{S} = \{\{u,m,n\} \in \hat{E}: u \in S\}.$$

For each $k \in \mathrm{N},\ k \geq 2,$ and each $\nu \in \{+,-\},$ let $\hat{S}^{\nu}_{k}$ be a subset of $\hat{S}$ given by

$$\hat{S}^{\nu}_{k} = \{\{u,m,n\} \in \hat{S}: u \in S^{\nu}_{k}\}.$$

3. Global bifurcation of solutions to problem (1)-(3)

Let $L: D(L) \subset \hat{E} \to C[0,\ 1] \oplus R^2$ be the linear operator defined by

$$L\hat{u} = L\{u,m,n\} = \{\ell(u), Tu(0), Tu(1)\},$$

where

$$D(L)=\{\hat{u}=\{u,m,n\}\in\hat{E}\,:\,u\in C^4[0,\,1],\;m=au(0),\;n=cu(1)\}$$

and $C[0,1]\oplus R^2$ has norm given by

$$\|\hat{u}\|_0=\|u\|_0+|m|+|n|.$$

Moreover, we define nonlinear operator $G:R\times\hat{E}\to C[0,1]\oplus R^2$ as follows:

$$G(\lambda,\hat{u})=G(\lambda,\{u(x),m,n\})=\{g(x,u(x),u'(x),u''(x),u'''(x),\lambda),0,0\},$$

where $m=au(0),\;\; n=cu(1)\}$. Then the nonlinear eigenvalue problem (1)-(3) reduces to the following problem

$$L\hat{u}=\lambda\hat{u}+\hat{G}(\lambda,\hat{u}),\;u\in D(L), \tag{8}$$

i.e., there is a one-to-one correspondence between the solutions of problems (1)-(3) and (8)

$$(\lambda,u)\leftrightarrow(\lambda,\hat{u})=(\lambda,\{u,m,n\}),m=au(0),n=cu(1). \tag{9}$$

Let $\varepsilon>0$ be an arbitrary fixed sufficiently small number. Since $\lambda_1=0$ we consider the following approximate problem

$$L\hat{u}+\varepsilon I\hat{u}=\lambda\hat{u}+\hat{G}(\lambda,\hat{u}),\;u\in D(L), \tag{10}$$

where $I:C[0,1]\oplus R^2\to C[0,1]\oplus R^2$ is the identity operator.

Note that the eigenvalues of the linear eigenvalue problem

$$L\hat{u}+\varepsilon I\hat{u}=\lambda\hat{u},\;u\in D(L). \tag{11}$$

are

$$\lambda_{k,\varepsilon}=\lambda_k+\varepsilon,\;k\in\mathrm{N},$$

and corresponding eigenfunctions are

$$u_{k,\varepsilon}(x)=u_{k,}(x),\;k\in\mathrm{N}.$$

Remark 1. It follows from [4, 6, 7] that for each $k\in\mathrm{N},k\geq 2,$

$$\hat{u}_k\in\hat{S}_k=\hat{S}_k^+\bigcup\hat{S}_k^-.$$

Let $L_\varepsilon=L+\varepsilon R.$ Since $\lambda_{1,\varepsilon}>0$ then it follows that there exists

$$L_\varepsilon^{-1}:C[0,1]\oplus R^2\to D(L_\varepsilon).$$

By following the argument in Lemma 3.3 of [9] we can show that that L_ε^{-1} is a continuous and compact map.

We introduce the following notations:

$$\hat{L}_\varepsilon=L_\varepsilon^{-1},\;\hat{G}_\varepsilon=L_\varepsilon^{-1}G.$$

Then the operator $\hat{G}_\varepsilon$ is completely continuous.

Note that problem (10) can be reduced to the following equivalent form

$$\hat{u} = \lambda \hat{L}_\varepsilon \hat{u} + \hat{G}_\varepsilon(\lambda, \hat{u}). \tag{12}$$

By condition (4) it follows from [16, Lemma 1] that

$$\hat{G}_\varepsilon(\lambda, \hat{u}) = o(\|\hat{u}\|) \text{ as } \|\hat{u}\| \to 0, \tag{13}$$

uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset R$.

In view of (13) by [14, Ch. 4, § 2, Theorem 2.1] the linearization of problem (12) at $\hat{u} = \hat{0}$ is the spectral problem

$$\hat{u} = \lambda \hat{L}_\varepsilon \hat{u}. \tag{14}$$

Note that problem (14) is equivalent to the spectral problem (11). Hence all characteristic values of the operator $\hat{L}_\varepsilon$ are nonnegative and simple.

Remark 2. It follows from [3, Lemma 1] that if $(\lambda, \hat{u}) \in R \times \hat{E}$ is a solution of problem (1)-(3) such that $\hat{u} \in \partial S_k^\nu, k \in \mathrm{N}, k \geq 2, \nu \in \{+,-\}$, then $\hat{u} = \hat{0}$, where $\hat{0} = \{0, 0, 0\}$.

Let $\hat{C}_\varepsilon$ the closure $R \times \hat{E}$ of the set of non-trivial solutions of (12).

We have the following unilateral global bifurcation result for nonlinear problem (12).

Theorem 1. For each $k \in \mathrm{N}, k \geq 2$, there exist continua $C_{k,\varepsilon}^+$ and $C_{k,\varepsilon}^-$ of the set $\hat{C}_\varepsilon$ which contain $(\lambda_{k,\varepsilon}, \hat{0})$, are contained in $(R \times \hat{S}_k^+) \bigcup \{(\lambda_{k,\varepsilon}, \hat{0})\}$ and $(R \times \hat{S}_k^-) \bigcup \{(\lambda_{k,\varepsilon}, \hat{0})\}$, respectively, and are unbounded in $R \times \hat{E}$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.3 from [19] using the above reasoning, Remarks 1 and 2, and condition (13).

Let $\hat{C}$ the closure $R \times \hat{E}$ of the set of non-trivial solutions of (8).

The following theorem is main result of this paper.

Theorem 2. For each $k \in \mathrm{N}, k \geq 2$, there exist continua $\hat{C}_k^+$ and $\hat{C}_k^-$ of the set $\hat{C}$ which contain $(\lambda_k, \hat{0})$, are contained in $(R \times \hat{S}_k^+) \bigcup \{(\lambda_k, \hat{0})\}$ and $(R \times \hat{S}_k^-) \bigcup \{(\lambda_k, \hat{0})\}$, respectively, and are unbounded in $R \times \hat{E}$.

Proof. Let $k \geq 2$ be arbitrary fixed natural number and let $\hat{Q}_k \in R \times \hat{E}$ be a bounded open set such that $(\lambda_k, \hat{0}) \in \hat{Q}_k$. Then for ε small, e.g. $0 < \varepsilon < \varepsilon_0$, we have $(\lambda_{k,\varepsilon}, \hat{0}) \in \hat{Q}_k$. Hence it follows from Theorem 1 that there exists a solution $(\lambda_\varepsilon, \hat{u}_\varepsilon)$ of problem (10) such that

$$(\lambda_\varepsilon, u_\varepsilon) \in \partial\hat{Q}_k \cap (R \times \hat{S}_k^\nu).$$

From equation

$$\ell(u) + \varepsilon u = \lambda y + g(x, y, y', y'', y''', \lambda), \ \ x \in (0, 1),$$

it is easy to see that the set of solutions $\{(\lambda_\varepsilon, u_\varepsilon) : 0 < \varepsilon < \varepsilon_0\}$ is precompact in $R \times E$, and consequently, $\{(\lambda_\varepsilon, \hat{u}_\varepsilon) : 0 < \varepsilon < \varepsilon_0\}$ is precompact in $R \times \hat{E}$.

Let $\{\varepsilon_r\}_{r=1}^\infty, : 0 < \varepsilon_r < \varepsilon_0$, be a sequence converging to 0. Then there exists a subsequence $\{\varepsilon_{r_l}\}_{lr=1}^\infty$ of a sequence $\{\varepsilon_r\}_{r=1}^\infty$ such that $\{(\lambda_{\varepsilon_{r_l}}, \hat{u}_{{r_l}\varepsilon}):\}_{l=1}^\infty$ converges in $R \times \hat{E}$ to a solution $(\lambda, \hat{u})$ of problem (8). It is obvious that

$$\partial\hat{Q}_k \cap \left(R \times \overline{\hat{S}_k^\nu}\right) = \partial\hat{Q}_k \cap (R \times (\hat{S}_k^\nu \cup \partial\hat{S}_k^\nu)).$$

If $\hat{u} \in \partial\hat{S}_k^\nu$, then it follows from Remark 2 that $\hat{u} = \hat{0}$ which is impossible since $\hat{Q}_k$ is a neighbourhood of $(\lambda_k, \hat{0})$. Consequently, we have

$$(\lambda, u) \in \hat{C} \cap \partial\hat{Q}_k \cap (R \times \hat{S}_k^\nu).$$

Now, due to the last relation, by [11, Theorem 2] we can easily obtain the statements of this theorem. The proof of Theorem 2 is complete.

Let C the closure in $R \times E$ of the set of non-trivial solutions of (1)-(3).

According to (9), from Theorem 2 we obtain the following result.

Theorem 3. For each $k \in \mathrm{N}, k \geq 2$, there exist continua C_k^+ and C_k^- of the set $\hat{C}$ which contain $(\lambda_k, 0)$, are contained in $(R \times S_k^+) \cup \{(\lambda_k, 0)\}$ and $(R \times S_k^-) \cup \{(\lambda_k, 0)\}$, respectively, and are unbounded in $R \times E$.

References

[**1**] Aliyev ZS. Some global results for nonlinear fourth order eigenvalue

problems. *Cent. Eur. J. Math.* **2014**, 12 (12), p. 1811-1828.

[**2**] Aliyev ZS. Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order. *Sb. Math.* **2016**, 207 (12), p.1625-1649.

[**3**] Aliyev ZS. Asadov XA. Global Bifurcation from zero in some fourth-order nonlinear eigenvalue problems. *Bull. Malays. Math. Sci. Soc.* **2021**, 44 (2), p. 981-992.

[**4**] Aliyev ZS, Guliyeva SB. Spectral properties for the equation of vibrating beam, *Proc. IMM, NAS Azerbaijan* **2015**, 41(1), p.135-145.

[**5**] Aliyev ZS, Ismayilova NA. Global bifurcation from zero in nonlinear Sturm-Liouville equation with a spectral parameter in the boundary condition, *Quaestiones Mathematicae* **2023**, 46 (11), p. 2233-2241.

[**6**] Aliyev ZS, Namazov FM. Spectral properties of a fourth-order eigenvalue problem with spectral parameter in the boundary conditions. *Electron. J. Differ. Equ.* **2017**, 2017(307), p. 1-11.

[**7**] Banks DO, Kurowski GJ. A Prufer transformation for the equation of a vibrating beam subject to axial forces. *J. Differential Equations* **1977**, 24(1), p. 57-74.

[**8**] Berestycki H. On some nonlinear Sturm-Liouville problems. *J. Differential Equations* **1977**, 26 (3), p. 375–390.

[**9**] Binding PA, Browne PJ, Watson BA. Spectral problem for nonlinear Sturm-Liouville equations with eigenparameter dependent boundary conditions. *Canad. J. Math.* **2000**, 52 (2), p. 248-264.

[**10**] Bolotin V.V. *Vibrations in Technique: Handbook in 6 Volumes, The Vibrations of Linear Systems, I.* Engineering Industry, Moscow ; 1978 [in Russian].

[**11**] Dancer EN. On the structure of solutions of nonlinear eigenvalue problems. *Indiana Univ. Math. J.* **1974**, 23 (11), p. 1069-1076.

[**12**] Fulton CT. Two-point boundary value problems with eigenparameter contained in the boundary conditions. *Proc. Roy. Soc. Edinburgh. Section A: Math.* **1977**, 77 (3-4), p. 293-308.

[**13**] J. B. Keller and S. Antman (eds.), *Bifurcation theory and nonlinear*

eigenvalue problems, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam; 1969.

[**14**] Krasnoselskii MA. *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Oxford; 1964.

[**15**] Lazer AC, McKenna PJ. Global bifurcation and a theorem of Tarantello. *J. Math. Anal. Appl.* **1994**, 181 (3), p. 648-655.

[**16**] Namazov FM. Bifurcation of solutions from zero to some nonlinear eigenvalue problems. *News of Baku State University, Ser. Phys.-Math. Sci.* **2023**, (2), p. 66-77.

[**17**] Przybycin J. Some applications of bifurcation theory to ordinary differential equations of the fourth order. *Ann. Polon. Math.* **1991**, 53 (2), p. 153–160.

[**18**] Rabinowitz PH. Nonlinear Sturm-Liouville problems for second order ordinary differential equations, *Commun. Pure Appl. Math.* **1970**, 23(6), p. 939-961.

[**19**] Rabinowitz PH. Some global results for nonlinear eigenvalue problems. *J. Funct. Anal.* **1971**, 7 (3), p. 487-513.

[**20**] Rynne BP. Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable. *J. Math. Anal. Appl.* **1998**, 228 (1), p. 141-156.