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ON STRONG SOLVABILITY OF ONE NONLOCAL BOUNDARY VALUE PROBLEM FOR POISSON EQUATION IN RECTANGLE

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Abstract

One nonlocal boundary value problem for the Poisson equation in an rectangular domain is considered in this work. The concept of strong solution of this problem is introduced. The correct solvability of this problem in weighted Sobolev spaces generated by a mixed norm is proved using the Fourier method. This is accomplished by exploiting the basis property of the system of eigenfunctions and associated functions of the corresponding spectral problem in weighted Lebesgue spaces, as well as the Hausdorff-Young inequality valid for this system.

Keywords: Poisson equation; nonlocal problem; weighted Sobolev space; strong solution

1. Introduction

Consider the following (formal for now) nonlocal boundary value problem for

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the Poisson equation:

$$u_{xx} + u_{yy} = f(x;y)\,, \quad 0 < x < 2\pi\,, \;\; 0 < y < h\,,$$
$$\left.\begin{array}{l} u(x,0) = \varphi(x), \;\; u(x,h) = \psi(x), \quad 0 < x < 2\pi, \\ u_x(0,y) = 0, \quad u(0,y) = u(2\pi,y), \quad 0 < y < h, \end{array}\right\} \tag{2}$$

In the case of a semi-infinite strip for one class of degenerate elliptic equation, which, in particular, covers the Laplace equation, this problem was previously considered by E.I. Moiseev [1] in the classical formulation. A similar problem was considered by M.E. Lerner and O.A. Repin [2]. The case of a semi-infinite strip for the Laplace equation is also considered in [3], where the strong solvability of a nonlocal boundary value problem in a weighted grand-Sobolev space is studied.

It is obvious that the solution to problem (1)-(2) can be sought as the sum of the solutions of the following two problems:

$$\left.\begin{array}{c} u_{xx} + u_{yy} = 0 \\ u(x,0) = \varphi(x), u(x,h) = \psi(x)\,, \\ u_x(0,y) = 0, u(0,y) = u(2\pi,y); \end{array}\right\} \tag{A}$$

and

$$\left.\begin{array}{c} u_{xx} + u_{yy} = f(x,y)\,, \\ u(x,0) = 0, u(x,h) = 0\,, \\ u_x(0,y) = 0, u(0,y) = u(2\pi,y). \end{array}\right\} \tag{B}$$

The strong solvability of Problem (A) in the domain $\Pi = (0,2\pi) \times (0,h)$ in weighted Sobolev spaces was considered in [4]. In this paper, we study the question of the strong solvability of Problem (B) in a weighted Sobolev space with a weight from the Muckenhoupt class, which, together with the result of [4], will allow us to formulate a well-posed solvability of Problem (1)–(2) in this space. The concept of a strong solution is defined, and the well-posed solvability of Problem (B) is proved by the spectral method.

Such problems have specific peculiarities compared to the ones with local conditions. Earlier, F.I.Frankl [5]; [6, p.453-456] considered the problem with nonlocal boundary condition for a mixed type equation. Bitsadze-Samarski problem [7] for elliptic equations is also nonlocal with supports on a part of the boundary of domain, and these supports are free of other boundary conditions. In [8], N.I.Ionkin and E.I.Moiseev solved the boundary value problem for multidimensional parabolic equations with nonlocal conditions, whose supports are the characteristic and the improper parts of the boundary of domain.

We note that the study of the solvability of elliptic equations with respect to the so-called non-standard function spaces, as well as in weighted Sobolev spaces, encounters certain difficulties compared to the weightless case. Therefore, the number of studies devoted to this area has been growing in recent years (see, for example, [3, 9 – 24]), and the development of the corresponding theory is far from complete. These attempts are also relevant in connection with the question of the basis properties of eigenfunctions and associated functions of spectral problems obtained by applying the method of separation of variables in the corresponding non-standard function spaces. The question of the basis property of the classical exponential system in weighted Lebesgue spaces was studied in [25, 26]. In this regard, we note the works [27 - 33], in which similar questions are studied and appropriate research methods are proposed.

2. Auxiliary concepts and facts

2.1. Notations.

We will use standard notations. N will be the set of positive integers, while $\alpha = (\alpha_1;\ \alpha_2) \in Z_+ \times Z_+$ will denote a multi-index, where $Z_+ = N \cup \{0\}$. Denote $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. By $|M|$ we will denote the Lebesgue measure of the set M; $\overline{M}$ will be the closure of M. $C^\infty(\overline{M})$ will stand for the infinitely differentiable functions on $\overline{M}$, and $C_0^\infty(M)$ will denote the infinitely differentiable and finite functions on M. Throughout this paper we will assume that p' is a conjugate number of p, $1 < p < +\infty$: $\frac{1}{p'} + \frac{1}{p} = 1$.

2.2. Weighted Sobolev space $W_{p,\nu}^2(\Pi)$.

Let's define our weighted Sobolev space. Let $\nu: [0, 2\pi] \to (0, +\infty)$ be some weight function, $\Pi = (0, 2\pi) \times (0, h)$. Denote by $L_{p;\nu}(\Pi)$ a Banach space of functions on Π with the mixed norm

$$\|f\|_{L_{p,\nu(\Pi)}} = \int_0^h \left(\int_0^{2\pi} |f(x;y)|^p \nu(x)dx \right)^{\frac{1}{p}} dy \ , \quad 1 < p < +\infty.$$

Denote by $W_{p,\nu}^2(\Pi)$ a Sobolev space generated by the norm

$$\|u\|_{W_{p;\nu}^2(\Pi)} = \sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L_{p,\nu}(\Pi)}.$$

Now denote by $L_{p,\nu}(I)$, where $I = (0, 2\pi)$, a weighted Lebesgue space generated by the norm

$$\|f\|_{L_{p,\nu}(I)} = \left(\int_I |f(x)|^p \nu(x)dx \right)^{\frac{1}{p}}.$$

We will also consider the weighted Sobolev space $W_{p,\nu}^2(I)$, generated by the norm

$$\|f\|_{W_{p,\nu}^2(I)} = \|f\|_{L_{p,\nu}(I)} + \|f'\|_{L_{p,\nu}(I)} + \|f''\|_{L_{p,\nu}(I)}.$$

We will need the class of Muckenhoupt weights $A_p(I)$ (see [26]). This is a class of $2\pi -$ periodic functions (i.e. the class of functions ν periodically extended to the whole of the axis with period 2π), satisfying the condition

$$\sup_{J\subset I} \left(\frac{1}{|J|} \int_J \nu(t)dt \right) \left(\frac{1}{|J|} \int_I |\nu(t)|^{-\frac{1}{p-1}} \right)^{p-1} < +\infty,$$

where sup is taken over all intervals $J \subset I$, and $|J|$ is a length of the interval J.

2.3. Statement of the problem.

For correct statement of considered boundary value problem we need firstly define the trace operator corresponding to the weighted Sobolev space $W_{p,\nu}^2(\Pi)$ on domain $\Pi = (0, 2\pi) \times (0, h) \subset R^2$ with boundary $\partial\Pi = I_0 \cup I_h \cup \Gamma_0 \cup \Gamma_{2\pi}$, where are denoted $I_0 = \{(x; 0) : \ 0 < x < 2\pi\}$, $I_h = \{(x; h) : 0 < x < 2\pi\}$, $\Gamma_0 = \{(0; y) : \ 0 < y < h\}$ and $\Gamma_{2\pi} = \{(2\pi; \ y): \ 0 < y < h\}$ .It is obvious that $L_{p,\nu}(I) \subset L_1(I)$.Then, the continuous embedding $W_{p,\nu}^2(\Pi) \ \subset \ W_1^2(\Pi)$ is also true. Consequently, every function $u \in W_{p,\nu}^2(\Pi)$ has traces $u|_{\partial\Pi}$ and $u_x|_{\partial\Pi}$ as functions of space $L_1(\partial\Pi; d\sigma)$ on the boundary (correctly defined with respect to the Lebesgue measure on $\partial\Pi$).

Consider the following nonlocal problem:

$$\Delta u = f(x;y)\,,\quad (x;y) \in \Pi, \tag{3}$$

$$u|_{I_0} = 0\,,\;\; u|_{I_h} = 0,\;\;\; u|_{\Gamma_0} = u|_{\Gamma_{2\pi}}\,,\;\; u_x|_{\Gamma_0} = 0, \tag{4}$$

where $f(x;y) \in L_{p,\nu}(\Pi)$. By the solution of this problem, we mean a function $u \in W^2_{p;\nu}(\Pi)$, which satisfies the equality (3) a.e. in Π and whose traces satisfy the relations (4) on the boundary $\partial\Pi$. Taking into account the above, we arrive at the following definition of strong solvability of problem (3)-(4).

Definition 2.1. *A function $u \in W^2_{p,\nu}(\Pi)$ is called a strong solution of the problem (3)-(4) if the equality (3) is satisfied for a.e. $(x;y) \in \Pi$ and its trace $u|_{\partial\Pi}$ satisfies the relations (4).*

2.4. Some facts from basis theory.

Let us present some well-known concepts and facts from basis theory. Let X be a Banach space. A system $\{u_n\}_{n\in N} \subset X$ is called a basis if any element $f \in X$ is uniquely represented as a series

$$f = \sum_{n=1}^{\infty} c_n\, u_n$$

convergent in the norm X. A system $\{u_n\}_{n\in N} \subset X$ is called complete in X if $\overline{\mathrm{Sp}}\{u_n\} = X$ and minimal in X if $u_n \notin \overline{\mathrm{Sp}}\{u_k\}_{k\neq n}$. It is known that each basis of the space X is a complete and minimal system in X, the converse is not true in general. The system $\{x_n\}_{n\in N} \subset X$ is complete in X $\Leftrightarrow$ $\varphi \in X^*$: $\langle x_n, \varphi\rangle = \varphi(x_n) = 0, \forall\, n\in N \Rightarrow \varphi = 0$. The system $\{u_n\}_{n\in N}$ is minimal in X if and only if there exists a biorthogonal system i.e. there exists a system $\{\vartheta_n\}_{n\in N} \subset X^*$ such that $\langle u_n, \vartheta_k\rangle = \vartheta_k(u_n) = \delta_{nk}$, where δ_{nk} is the Kronecker symbol.

Basis criterion. *The system $\{u_n\}_{n\in N} \subset X$ is a basis of the space X if and only if the following conditions are satisfied:*

1) $\{u_n\}_{n\in N}$ is complete and minimal in X;

2) The projectors

$$P_n f = \textstyle\sum_{k=1}^{n} \langle f, \vartheta_k\rangle\, u_k\,,$$

where $\{\vartheta_k\}_{k\in N}$ is a biorthogonal system, are uniformly bounded.

Definition 2.2. *A system $\{u_n\}_{n\in N} \subset X$ is called a basis with brackets in X if there exists a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$, such that each element of $f \in X$ is uniquely represented as a series*

$$f = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i \, u_i \, ,$$

convergent in the norm X .

Basis criterion with brackets. *The system $\{u_n\}_{n\in N} \subset X$ is a basis with brackets of the space X if and only if the following conditions are satisfied:*

1) $\{u_n\}_{n\in N}$ is complete and minimal in X;

2) The projectors

$$P_{n_k} f = \sum_{i=1}^{n_k} \langle f, \vartheta_i \rangle \, u_i \, ,$$

where $\{\vartheta_n\}_{n\in N}$ is a biorthogonal system, are uniformly bounded.

Proposition 2.1. ([27]) *Let the system $\{u_n\}_{n\in N}$ form a basis with brackets in X and be uniformly minimal, and let the sequence $\{n_{k+1} - n_k\}_{k\in N}$ be bounded. Then the system $\{u_n\}_{n\in N}$ forms an usually basis in X.*

2.5. Basicity of root functions.

Applying the Fourier method to solution of the problem (B) (or problem (3)-(4)) leads to the following spectral problem

$$\left. \begin{array}{c} u''(x) + \lambda u(x) = 0, x \in I, \\ u(0) = u(2\pi), u'(0) = 0. \end{array} \right\} \tag{5}$$

The eigenvalues of problem (5) are $\lambda_n = n^2$, $n \in Z_+$, and corresponding eigen functions are $u_{2n}(x) = \cos nx$, $n \in Z_+$. Each eigenfunction $u_{2n}(x), n \in N$, has one associated function $u_{2n-1}(x) = x \sin nx$, $n \in N$. Consider the collection of root functions

$$u_0(x) = 1, u_{2n}(x) = \cos nx, u_{2n-1}(x) = x \sin nx, \; n \in N, \tag{6}$$

and also set

$$\vartheta_0(x) = \frac{1}{2\pi^2}(2\pi - x), \; \vartheta_{2n}(x) = \frac{1}{\pi^2}(2\pi - x)\cos nx,$$
$$\vartheta_{2n-1}(x) = \frac{1}{\pi^2}\sin nx, \; n \in N. \tag{7}$$

Note that these systems are biorthogonal conjugate, which can be verified directly. To obtain our main result, we will significantly use the following theorem.

Theorem 2.3 *Let $\nu \in A_p(I)$, $1 < p < +\infty$. Then the system (6) forms a basis for $L_{p,\nu}(I)$.*

Proof. To prove the theorem, we first apply the method of work [34]. Estimating the projections $P_{2n}, n \in Z_+$, in a weighted space reduces to estimating the integrals of the Green's function over an expanding system of contours $\Gamma_n = \left\{\rho \in C\colon |\rho| = n + \frac{1}{2},\ -\frac{\pi}{2} \le \arg\rho \le \frac{\pi}{2}\right\}$ in the complex $\rho -$ plane, which in turn leads to estimating integrals of the Hilbert transform type in a weighted Lebesgue space. Using the boundedness of the Hilbert transform in a weighted Lebesgue space with a weight from the Muckenhoupt class [25], we obtain the uniform boundedness of the system of projectors $\{P_{2n}\}_{n \in Z_+}$ which, according to the basis criterion with brackets, means that system (6) is a basis with brackets in $L_{p,\nu}(I)$. On the other hand, it is easy to see that the norms of the functions from (6), as well as from (7), are uniformly bounded. Then, applying Proposition 2.1, we obtain that system (6) forms an usually basis in $L_{p,\nu}(I)$.

3. Main Results

In this section, we will study the existence and uniqueness of strong solution of the problem (B) in the sense of Definition 2.1. For $f(x, y) \in L_{p,\nu}(\Pi)$ denote $F_n(y) = \langle f(\cdot, y), \vartheta_n(\cdot)\rangle$, i.e.

$$\left.\begin{aligned} F_0(y) &= \frac{1}{2\pi^2}\int_0^{2\pi} f(x,y)(2\pi - x)\,dx, \\ F_{2n}(y) &= \frac{1}{\pi^2}\int_0^{2\pi} f(x,y)(2\pi - x)\cos nx\,dx, \\ F_{2n-1}(y) &= \frac{1}{\pi^2}\int_0^{2\pi} f(x,y)\sin nx\,dx, n \in N. \end{aligned}\right\} \tag{8}$$

The main result of the work is the following theorem.

Theorem 3.1 *Let the weight function $\nu(x)$ belong to the class $A_p(I)$, $1 < p < +\infty$, and the function $f(x, y) \in W^{1,0}_{p,\nu}(\Pi)$ and satisfies the following condition $f(0, y) = f(2\pi, y)$. Then problem (B) has a unique solution in $W^2_{p,\nu}(\Pi)$ and moreover it is valid the following estimate*

$$\|u\|_{W^2_{p;\nu}(\Pi)} \le c\|f\|_{W^{1,0}_{p,\nu}(\Pi)}$$

where $c > 0$ is a constant independent of $f(x, y)$.

Proof. First of all,we note that the uniqueness of the solution follows from the results of $[4]$.Therefore,we proceed directly to the proof of the existence of a solution.Suppose $u(x,y)\in W^2_{p;\nu}(\Pi)$ is a solution of the problem (3)-(4). Consider $U_n(y)=\langle u(\cdot,y),\vartheta_n(\cdot)\rangle$, i.e.

$$\left.\begin{aligned}
&U_0(y)=\frac{1}{2\pi^2}\int_0^{2\pi} u(x,y)\,(2\pi-x)\,dx,\\
&U_{2n}(y)=\frac{1}{\pi^2}\int_0^{2\pi} u(x,y)\,(2\pi-x)\cos nx\,dx,\\
&U_{2n-1}(y)=\frac{1}{\pi^2}\int_0^{2\pi} u(x,y)\,\sin nx\,dx\ ,\ \ n\in N.
\end{aligned}\right\}\qquad(9)$$

It is absolutely clear that for a.e. $x\in I$ the relation

$$u(x,y+\delta)-u(x,y)=\int_y^{y+\delta}\frac{\partial u(x,t)}{\partial t}dt,\quad \forall\, y\in(0.h),\qquad(10)$$

holds. Since $\frac{\partial u}{\partial y}\in L_1(\Pi)$, applying Theorem 1.1.1 of [35, p.13], from (10) we obtain that the functions $U_n(y)$ are twice differentiable and can be differentiated under the integral sign. Since the function $u(x,y)$ satisfies the equation (3), multiplying it by $\sin nx$ (by $(2\pi-x)\cos nx$) and integrating over I, we obtain the following relations for $U_{2n-1}(y)$ (respectively, for $U_{2n}(y)$):

$$\left.\begin{aligned}
&U_0''(y)=F_0(y), && y\in(0,h)\\
&U''_{2n-1}(y)-n^2U_{2n-1}(y)=F_{2n-1}(y), && y\in(0,h),\\
&U''_{2n}(y)-n^2U_{2n}(y)=-2nU_{2n-1}(y)+F_{2n}(y)\,, && y\in(0,h).
\end{aligned}\right\}\qquad(11)$$

where the functions F_k(y), k$\in Z_+$ are defined by formulas (8).

Applying the Newton-Leibniz formula and taking into the account the first boundary condition (4),we obtain

$$u_\xi(x):=u(x,\xi)=u(x,0)+\int_0^\xi\frac{\partial u(x,y)}{\partial y}\,dy=\int_0^\xi\frac{\partial u(x,y)}{\partial\xi}\,d\xi,\ \ a.e.\ \ x\in I.$$

Consequently,

$$|u(x,\xi)|\le\int_0^\xi\left|\frac{\partial u(x,y)}{\partial y}\right|dy,\qquad a.e.\ \ x\in I.$$

Hence it immediately follows that

$$\left\|u_\xi\right\|_{L_1(I)}=\int_I|u(x,\xi)|\,dx\le\int_I\int_0^\xi\left|\frac{\partial u(x,y)}{\partial y}\right|dy\,dx.\qquad(12)$$

We have $\left|\Pi_\xi\right|\to 0$ as $\xi\to+0$. Then from (12) it follows that

$$u_\xi(\cdot) \to 0\,,\ \xi \to +0, \tag{13}$$

in the norm of the space $L_1(I)$.

Similarly, taking into account the second boundary condition (4), we have

$$u(x,\xi) = u(x,h) - \int_\xi^h \frac{\partial u(x,y)}{\partial y}\,dy = -\int_\xi^h \frac{\partial u(x,y)}{\partial y}\,dy,\ \ a.e.\ \ x \in I.$$

Hence,

$$\int_I |u(x,\xi)|\,dx \le \int_I \int_\xi^h \left|\frac{\partial u(x,y)}{\partial y}\right| dy\,dx. \tag{14}$$

As $\left|\Pi\backslash\Pi_\xi\right| \to 0$ when $\xi \to h-0$, from (14) it follows that

$$u_\xi(\cdot) \to 0\,,\ \xi \to h-0, \tag{15}$$

in the norm of the space $L_1(I)$.

On the other hand, it is clear that $U_n(y) \in W_1^2(0,h)$. Hence it immediately follows that there exist the limits

$$\lim_{y\to+0} U_n(y) = U_n(0)\,, \qquad \lim_{y\to h-0} U_n(y) = U_n(h), \quad \forall\, n \in Z^+.$$

By (13) and (15), from the last two relations it immediately follows that

$$U_n(0) = 0, \quad U_n(h) = 0, \quad \forall\ n \in Z_+, \tag{16}$$

Solution of the problem (11), (16) in the case of odd indices is

$$U_{2n-1}(y) = -\frac{1}{n}\frac{\sinh n(h-y)}{\sinh nh}\int_0^y \sinh nt\, F_{2n-1}(t)dt -$$

$$-\frac{1}{n}\frac{\sinh ny}{\sinh nh}\int_y^h \sinh n(h-t) F_{2n-1}(t),\ \forall n \in N, \tag{17}$$

and the solution of the problem (11), (16) in the case of even indices is

$$U_0(y) = -\frac{y}{h}\int_0^h (h-t)F_0(t)dt + \int_0^y (y-t)F_0(t)dt, \tag{18}$$

$$U_{2n}(y) =$$

$$-\frac{1}{n}\frac{\sinh n(h-y)}{sinh\,n\,h}\int_0^y F_{2n}(t)\sinh nt\,dt - \frac{1}{n}\frac{\sinh ny}{\sinh nh}\int_0^y F_{2n}(t)\sinh nt\,dt -$$

$$-\frac{2}{n}\frac{\sinh n(h-y)}{\sinh nh}\int_0^y \frac{\sinh n(h-t)}{\sinh nh}\int_0^t F_{2n-1}(\tau)\sinh n\tau\,d\tau dt -$$

$$-\frac{2}{n}\frac{\sinh n(h-y)}{\sinh nh}\int_0^y \frac{\sinh nt}{\sinh nh}\int_t^h F_{2n-1}(\tau)\sinh n(h-\tau)\,d\tau dt\; -$$

$$-\frac{2}{n}\frac{\sinh ny}{\sinh nh}\int_y^h \frac{\sinh n(h-t)}{\sinh nh}\int_0^t F_{2n-1}(\tau)\sinh n\tau\,d\tau dt\; -$$

$$-\frac{2}{n}\frac{\sinh ny}{\sinh nh}\int_y^h \frac{\sinh nt}{\sinh nh}\int_t^h F_{2n-1}(\tau)\sinh n(h-\tau) d\tau dt, \quad \forall n\in N. \quad (19)$$

Consider the function

$$u(x,y)=U_0(y)+\sum_{n=1}^{\infty} U_n(y)\,u_n(x)=U_0(y)+$$

$$+\sum_{k=1}^{\infty}\left(U_{2k}(y)\cos kx+U_{2k-1}(y)\,x\sin kx\right)\;,\;(x,y)\in \Pi, \quad (20)$$

where the coefficients $U_0(y)$, $U_{2k}(\cdot)$, $U_{2k-1}(\cdot)$, $k\in N$, are defined by (17)-(19). Let's show that the function $u(x,y)$ belongs to $W^2_{p;\nu}(\Pi)$. Denote by $u_{\alpha_1,\alpha_2}(x,y)$ the sum of the series obtained by the formal differentiation of the series (20), i.e.

$$u_{\alpha_1,\alpha_2}(x,y)=U_0^{(\alpha_2)}(y)+\sum_{n=1}^{\infty} U_n^{(\alpha_2)}(y)\,u_n^{(\alpha_1)}(x), \quad (21)$$

where $\alpha_1,\alpha_2\in Z^+, \alpha_1+\alpha_2=0,1,2; u_{0,0}(x,y)=u(x,y)$.

Let's first consider the series

$$u_1(x,y)=\sum_{k=1}^{\infty} U_{2k-1}(y)\,x\sin kx.$$

Differentiating this series formally term-by-term, we have

$$\frac{\partial^2 u_1}{\partial y^2}=\sum_{k=1}^{\infty} U''_{2k-1}(y)\,x\sin kx=\sum_{k=1}^{\infty} k^2U_{2k-1}(y)\,x\sin kx,$$

$$\frac{\partial u_1}{\partial x}=\sum_{k=1}^{\infty} U_{2k-1}(y)\sin kx+\sum_{k=1}^{\infty} kU_{2k-1}(y)\,x\cos kx,$$

$$\frac{\partial^2 u_1}{\partial x^2}=2\sum_{k=1}^{\infty} kU_{2k-1}(y)\cos kx-\sum_{k=1}^{\infty} k^2U_{2k-1}(y)\,x\sin kx. \quad (24)$$

Denote

$$w(x,y)=\sum_{k=1}^{\infty} k^2U_{2k-1}(y)\,x\sin kx.$$

Let's show that the function $w(x,y)$ belongs to $L_{p;\nu}(\Pi)$.

From (7), integrating by parts, we obtain

$$F_{2k-1}=\frac{1}{\pi^2}\int_0^{2\pi} f(x;y)\sin kx\,dx=-\frac{1}{\pi^2 k}\big(f(2\pi;y)-f(0;y)\big)+$$

$$+\frac{1}{\pi^2 k}\int_0^{2\pi} f_x'(x;y)\cos kx\,dx=\frac{1}{\pi^2 k}\int_0^{2\pi} f_x'(x;y)\cos kx\,dx=\frac{1}{k}F_{2k}'(\mathrm{y})$$

where is denoted by

$$F_{2k}'(\mathrm{y})=\frac{1}{\pi^2}\int_0^{2\pi} f_x'(x;y)\cos kx\,dx.$$

It is known that if $\nu\in A_p(I)$, then $\exists\ \alpha>1$: $\nu\in L_\alpha(I)$ (see, e.g., [36, p. 395]). Let $\frac{1}{\alpha}+\frac{1}{\alpha\prime}=1$. Applying Holder's inequality, we obtain

$$\left(\int_0^{2\pi}|w(x,y)|^p\nu(x)dx\right)^{\frac{1}{p}}\le\left(\int_0^{2\pi}\nu^\alpha(x)dx\right)^{\frac{1}{\alpha}}\left(\int_0^{2\pi}|w(x,y)|^{p\alpha\prime}dx\right)^{\frac{1}{p\alpha\prime}}$$

$$=c\left(\int_0^{2\pi}|w(x,y)|^{p_1}dx\right)^{\frac{1}{p_1}},$$

where $c=\left(\int_0^{2\pi}\nu^\alpha(x)dx\right)^{\frac{1}{\alpha}}$ (consequently, does not depend on $w\,(x,y)$) and $p_1=p\alpha'$. Let us consider seperately the cases $p\ge 2$ and $1<p<2$.

Let first $p\ge 2$. Then $p_1=p\alpha'>2$. From the previous inequality we have

$$\left(\int_0^{2\pi}|w(x,y)|^p\nu(x)dx\right)^{\frac{1}{p}}\le c\sum_{k=1}^{\infty}|U_{2k-1}(y)|\left(\int_0^{2\pi}|u_{2k-1}(x)|^{p_1}dx\right)^{\frac{1}{p_1}}\le$$

$$\le \mathrm{c}_1\sum_{\mathrm{k}=1}^{\infty}k^2|U_{2k-1}(y)|\le$$

$$\le \mathrm{c}_1\sum_{\mathrm{k}=1}^{\infty}k^2\left|-\frac{1}{k}\frac{\sinh k(h-y)}{\sinh kh}\int_0^y\sinh kt\,F_{2k-1}(t)dt\right.$$

$$\left.-\frac{1}{k}\frac{\sinh ky}{\sinh kh}\int_y^h\sinh k(h-t)F_{2k-1}(t)\right|\le$$

$$\leq c_1 \sum_{k=1}^{\infty}\left(\frac{k}{\sinh kh}\left|\left(\sinh k(h-y)\int_0^y \sinh kt\, F_{2k-1}(t)dt)dy\right)\right.\right.$$
$$\left.\left.+\left(\sinh ky \int_y^h \sinh k(h-t)F_{2k-1}(t)dt\right)dy\right|\right) \leq$$

$$\leq \sum_{k=1}^{\infty}\frac{k}{\sinh kh}\left(\left(\sinh k(h-y)\int_0^y \sinh kt\,|F_{2k-1}(t)|dt)dy\right)\right.$$
$$\left.+\left(\sinh ky \int_y^h \sinh k(h-t)|F_{2k-1}(t)|dt\right)\right)dy$$

Hence, first integrating with respect to $y \in (0,h)$ and then applying Holder's inequality for any $\beta \in (1,\infty)$, we obtain

$$\|w\|_{L_{p,\nu}(\Pi)} \leq c_1 \sum_{k=1}^{\infty}\left(\frac{k}{\sinh kh}\left(\int_0^h (\sinh k(h-y)\int_0^y \sinh kt\,|F_{2k-1}(t)|dt)dy\right)\right.$$
$$\left.+\left(\int_0^h \sinh ky \int_y^h \sinh k(h-t)|F_{2k-1}(t)|dt\right)dy\right) \leq$$
$$\leq c_1 \sum_{k=1}^{\infty}\int_0^h |F_{2k-1}(t)|dt \leq c_1 \int_0^h \left(\sum_{k=1}^{\infty}|F_{2k-1}(t)|\right) dt \leq$$
$$\leq c_2 \int_0^h \left(\sum_{k=1}^{\infty}\frac{1}{k}|F'_{2k}(t)|\right)dt \leq c_2 \left(\sum_{k=1}^{\infty}\frac{1}{k^{\beta'}}\right)^{\frac{1}{\beta'}} \int_0^h \left(\left(\sum_{k=1}^{\infty}|F'_{2k}(t)|^{\beta}\right)^{\frac{1}{\beta}}\right)dt$$

Now, assuming $\beta \geq 2$ and applying classical Hausdorff-Young inequality (see, e.g. [37, p.154]), we obtain

$$\|w\|_{L_{p,\nu}(\Pi)} \leq c_3 \int_0^h \left(\left\|\frac{\partial f}{\partial x}\right\|_{L_{\beta'}(I)}\right)dt. \tag{25}$$

It is known (see, e.g. [25,26]) that if $\nu \in A_p(I)$, $1 < p < +\infty$, then $\exists\, q$: $1 < q < p \Rightarrow \nu \in A_q(I)$. Let $r = \frac{p}{q}$ and $g \in L_{p,\nu}(I)$. Then $1 < r < p$ and we have

$$\left(\int_I |g|^r dx\right)^{\frac{1}{r}} = \left(\int_I |g|^{\frac{p}{q}} \nu^{\frac{1}{q}} \nu^{-\frac{1}{q}} dx\right)^{\frac{1}{r}} \le \left(\int_I |g|^p \nu\, dx\right)^{\frac{1}{qr}} \left(\int_I \nu^{-\frac{q'}{q}}\, dx\right)^{\frac{1}{q'r}} =$$
$$= \left(\int_I |g|^p \nu\, dx\right)^{\frac{1}{p}} \left(\int_I \nu^{-\frac{1}{q-1}}\, dx\right)^{\frac{q-1}{p}}.$$

As $-\frac{q'}{q} = \frac{1}{1-q}$, from $\nu \in A_q(I)$ it follows that $\nu^{-\frac{1}{q-1}} \in L_1(I)$. Then, the last inequality means $g \in L_r(I)$ and

$$\|g\|_{L_r(I)} \le c\|g\|_{L_{p;\nu}(I)},$$

where $c > 0$ is a constant independent of g. Also note that the continuous embedding $L_{p,\nu}(I) \subset L_\alpha(I)$ is true for every $\alpha \in (1, r)$. Let's choose β big enough to satisfy the condition $1 < \beta' < r$. Then from (25) we obtain

$$\|w\|_{L_{p;\nu}(\Pi)} \le c\left(\left\|\frac{\partial f}{\partial x}\right\|_{L_{p,\nu}(\Pi)}\right)$$

Now let $p \in (1,2)$. As in the previous case, note that there exists a number $\alpha > 1$ such that $\nu \in L_\alpha(I)$. But then $\nu \in L_s(I)$ for every $s \in (1, \alpha)$. Therefore, choosing $\alpha > 1$ close enough to 1, we can provide that $p_1 = p\,\alpha' > 2$ (this is possible, because $\alpha' \to +\infty$ as $\alpha \to 1 + 0$). With this, further considerations are carried out similar to the previous case.

Other series from (21)-(23), and, consequently, all series from (20) are estimated in a similar way. So, as a result, we obtain

$$\|u\|_{W^2_{p;\nu}(\Pi)} \le c\left(\left\|\frac{\partial f}{\partial x}\right\|_{L_{p,\nu}(\Pi)}\right),$$

where $c > 0$ is a constant independent of f. The fulfillment of equation (3) for u(·;·) is verified directly. The fulfillment of boundary conditions (4) follows from the relations for the trace of this function, proved in our previous work [4].

The theorem is proved.

Thus, based on Theorem 3.1 of this article and our previous work [4], we obtain the validity of the following theorem on the correct solvability of problem (1)-(2).

Theorem 3.2 *Let the weight function $\nu(x)$ belong to the class $A_p(I), 1 <$*

$p < \infty$ *and, and the boundary functions* $\varphi(x)$ *and* $\psi(x)$ *belong to* $W^2_{p;\nu}(I)$ *and satisfy the following conditions*

$$\varphi(0) - \varphi(2\pi) = \varphi'(0) = \psi(0) - \psi(2\pi) = \psi'(0) = 0,$$

and the function $f(x,y) \in W^{1,0}_{p,\nu}(\Pi)$ *and satisfies the condition* $f(0,y) = f(2\pi, y)$. *Then problem* (1)-(2) *has a unique solution in* $W^2_{p,\nu}(\Pi)$ *and moreover it is valid the following estimate*

$$\|u\|_{W^2_{p;\nu}(\Pi)} \le c\left(\|\varphi\|_{W^2_{p;\nu}(I)} + \|\psi\|_{W^2_{p;\nu}(I)} + \|f\|_{W^{1,0}_{p,\nu}(\Pi)}\right),$$

where $c > 0$ *is a constant independent of* φ, ψ *and* f.

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References

[1] Moiseev EI. Solving one nonlocal boundary value problem by spectral method. Diff. uravn. 1999, 35(8), 1094-1100. (in Russian)

[2] Lerner ME, Repin OA. On Frankl-type problems for some elliptic equations with different kinds of degeneration. Diff. uravn. 35(8), 1999, 1087-1093. (in Russian)

[3] Mammadov TJ. Strong solvability of a nonlocal problem for the Laplace equation in weighted grand Sobolev spaces, Azerbaijan Journal of Mathematics, 13(1), 2023, 188-204.

[4] T.Gasymov, B.Akhmadli, Ü.Yıldız. On strong solvability of one nonlocal boundary value problem equation in rectangle. Turkish Journal of Mathematics, 48(1), 2024, 21-33.

[5] Frankl FI. Unterschallströmungen um Profile mit Überschallzonen, die mit geradem Verdichtungsstoß enden. Prikl. matematika i mexanika, 20(2), 1956, 196-202. (in Russian)

[6] Frankl FI. Selected works on gas dynamics. Moscow, 1973, 711 p. (in Russian)

[7] Bitsadze AV, Samarski AA. On some simplest generalizations of linear elliptic boundary value problems. Dokl. AN SSSR, 185(4), 1969, 739-740. (in Russian)

[8] Ionkin NI, Moiseev EI. On the problem for heat equation with two-point boundary conditions. Diff. uravn. 15(7), 1979, 1284-1295. (in Russian)

[9] Bilalov BT, Ahmadov TM, Zeren Y, Sadigova SR. Solution in the Small and Interior Schauder-type Estimate for the m-th Order Elliptic Operator in Morrey-Sobolev Spaces. Azerbaijan Journal of Mathematics, 12(2), 2022, 190-219.

[10] Bilalov BT, Ahmedzadeh NR, Garayev TZ. Some Remarks on Solvability of Dirichlet Problem for Laplace Equation in Non-standard Function Spaces, Mediterranean Journal of Mathematics, 19(133), 2022, 25 p.

[11] Bilalov BT, Sadigova SR. On solvability in the small of higher order elliptic equations in grand-Sobolev spaces. Complex Variables and Elliptic Equations, 63(3), 2021, 2117-2130. doi: 10.1080/17476933.2020.1807965

[12] Bilalov BT, Sadigova SR. Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces. Sahand Communications in Mathematical Analysis, 1(2), 2021, 129-148. doi: 10.22130/SCMA.2021.521544.893

[13] Bilalov BT, Sadigova SR. On local solvability of higher order elliptic equations in rearrangement invariant spaces. Sibirskii Matematicheskii Zhurnal, 63(3), 2022, 516–530.doi: 10.33048/smzh.2022.63.304

[14] Bilalov BT, Sadigova SR. On the Fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces. Ricerche di Matematica, 2021.doi: 10.1007/s11587-021-00599-9.

[15] Bilalov BT, Zeren Y, Sadigova SR, Seyma C. Solvability in the small of m-th order elliptic equations in weighted grand-Sobolev spaces. Turkish. J. Math. 46, 2022, 2078 –2095.doi:10.55730/1300-0098.3255

[16] Bui TQ, Bui TA, Duong XT. Global regularity estimates for non-divergence elliptic equations on weighted variable Lebesgue spaces, Communications in Contemprory Math. 2020, 2050014. doi: 10.1142/S0219199720500145.

[17] Byun SS, Lee M. On weighted estimates for elliptic equations with BMO coefficients in non-divergence form. International Journal of Mathematics, 26(1) 2015, 1550001. doi: 10.1142/ S0129167X15500019.

[18] Caso L, D'Ambrosio R, Softova L. Generalized Morrey Spaces over Unbounded Domains, Azerbaijan Journal of Mathematics, 10(1), 2020, 193-208.

[19] Castillo RE, Rafeiro H, Rojas EM. Unique Continuation of the Quasilinear Elliptic Equation on Lebesgue Spaces L_p. Azerbaijan Journal of Mathematics, 11(1), 2021, 136-153.

[20] Coifman RR, Fefferman C. Weighted norm inequalities for maximal functions and singular integrals. Studia Mathematica, LI, 1974, 241-250.

[21] Dong H, Kim D. On L_p-estimates for elliptic and parabolic equations with A_p weights. Trans. Amer. Math. Soc. 370(7), 2018, 5081-5130. doi: 10.1090/tran/7161

[22] Dynkin EM, Osilenker VR. Weighted estimates of singular integrals and their applications. Matematicheskiy analiz, Moscow, VINITI, 21, 1983, 42-129. (in Russian)

[23] Palagachev DK, Softova LG. Elliptic Systems in Generalized Morrey Spaces. Azerbaijan Journal of Mathematics, 11 (2), 2021, 153-162.

[24] Gasymov, T.B., Akhmadli, B.Q.: On strong solvability of one nonlocal boundary value problem for Laplace equation in Grand Sobolev space in rectangle. Caspian Journal of Applied Mathematics, Ecology and Economics. 12(1), 3-15 (2024).

[25] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165, 207-226 (1972)

[26] R. A. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176, 227-252 (1973).

[27] Bilalov BT, Gasymov TB, Maharramova GV. Basis property of eigenfunctions in Lebesgue spaces for a spectral problem with a point of discontinuity. Differential Equations, 55(12), 2019, 1544-1553. doi:10.1134/S0012266119120024

[28] Bilalov BT, Gasymov TB, Maharramova GV, On basicity of eigenfunctions of one discontinuous spectral problem in Morrey type spaces, The Aligarh

Bulletin of Mathematics, 35(1-2), 119-129 (2016).

[29] Gasymov TB, Maharramova GV, Jabrailova A.N., Spectral properties of the problem of vibration of a loaded string in Morrey type spaces, Proc. of the IMM of NASA, Azerbaijan, v.44, no1, 116-122 (2018).

[30] Gasymov TB, Akhtyamov AM, Ahmedzade NR. On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces. Proc. of the IMM of NASA, Azerbaijan, 46(1), 2020, 32-44. doi: 10.29228/proc.15

[31] Gasymov TB, Guliyeva AA. On basicity of the system of exponents with linear phase in Morrey-Lebesgue space. Advances in Analysis, 3(2), 2018, 113-120. doi:10.22606/ aan2018.32005

[32] Gasymov TB, Hashimov ChM. On atomic decomposition in Banach spaces. Sahand Communications in Mathematical Analysis, 9(1), 2018, 15-32. doi: 10.22130/scma.2018.22984

[33] Sirin F, Zeren Y, Gasymov T. On basicity of eigenfunctions of one discontinuous differential operator in Banach function spaces. Authorea, November 02, 2022, 1-30. doi: 10.22541/au.166736702.23017975/v1

[34] Benzinger HE. L^p behavior of eigenfunction expansions. Trans. Amer. Math. Soc. 174, 1972, 333-344.

[35] Mikhlin SG. Linear partial equations. Moscow, 1977, 432 p. (in Russian)

[36] Garcia-Cuerva J, Rubio De Francia JL. Weighted Norm Inequalities and Related Topics, North-Holland, 1985, 615 p.

[37] Zygmund A. Trigonometric series. Moscow, 1965, 538 p. (in Russian)