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THE REPRESENTATION OF GENERALIZED WEIGHT HOLDER H_ω^p SPACES AS THE SUM OF BANACH SPACES

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Abstract

In this work, the representation of generalized weight Hölder spaces H_ω^p as the sum of Banach spaces is given and the criterion of coincidence of suitable pairs with each other is found.

Keywords: *Hölder spaces, weight functions, weighted evaluations.*

Mathematics Subject Classification (2020): *34B05, 34B08, 34C10, 34C23, 34L10, 47A75, 74H45*

1. Introduction

Lets assume that $\rho_1(x)$ and $\rho_2(x)$, $x \in [0,l]$, $l > 0$ are continuous functions and $\rho_1(0) = \rho_1(l) = 0,\ \rho_1(x) > 0,\ \ \rho_2(x) > 0,\ \ x \in [0,l)$. Now define $\rho(x) = \rho_1(x) \cdot \rho_2(x)$. For simplicity, we will replace $\rho_2(l-x)$ by $\rho_2(x)$.

Definition 1. If there exist increasing functions ρ_1^*, ρ_2^*, then:

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a) $\rho_i^*(2\xi) \sim \rho_i^*(\xi)$, $i = 1,2$;

b) $\rho_1(x) \sim \rho_1^*(x)$, $\rho_2(x) \sim \rho_2^*(l-x)$

If these relations hold, then we say $\rho \in \mathrm{P}$.

Definition 2. Suppose that $\rho_i(x)$ $(i = 1,2)$ non-decreasing functions and $\rho_i(0) = 0$; $\rho(x) = \rho_1(x)\rho_2(l-x)$, $x \in [0,l]$, $\omega(b)$, $\delta \in (0,l]$ is the modul of continuousness.

The space

$$H_\omega^p = \left\{ u \in C(0,l) : \lim_{x\to 0}(\rho u)(x) = \lim_{x\to l}(\rho u)(x) = 0 \right\}$$

is the generalized Hölder space.

The norm in this space is defined by the following equation:

$$\|u\| = \sup_{\substack{x_1,x_2\in[0,l]\\ x_1\neq x_2}} \left(\left|(\rho u)(x_1) - (\rho u)(x_1)\right| / \omega(|x_1 - x_2|) \right).$$

Let define the set of functions $f \in C(0,l]$ $(f \in C[0,l))$ by $H_\omega^{p_1}(0,l]$ $(H_\omega^{\rho_2}[0,l))$ so that,

$$f(t) = \frac{\varphi(t)}{\rho_1(t)} \quad \left(f(t) = \frac{\varphi(t)}{\rho_2(t)} \right),$$

here

$\varphi \in H_\omega[0,l]$, $\varphi(0) = 0$ $(\varphi(l) = 0)$.

And also define the set of functions $f \in C(0,l)$ by $H_\omega^{\rho_1\rho_2}$, so that

$$f(t) = \varphi(t) / \rho_1(t) \cdot \rho_2(t).$$

Definition 3. Lets define the set of functions $\rho \in \mathrm{P}$ by P_0, so that

$$\exists x^* \in (0,l), \exists r > 0, \ \forall x \in (x^* - r, x^* + r) \left|\rho(x) - \rho(x^*)\right| \leq const\,\omega(|x - x^*|)$$

2.Theorem about the representation of Banach spaces of weighted generalized Hölder spaces H_ω^ρ

Theorem 1. Suppose that $\rho = \rho_1\rho_2 \in \mathrm{P}_0$ and

$$\tilde{\rho}_i(t)=\begin{cases}\rho_1(t)\rho_2(t)/\rho_i\left(\frac{l}{2}\right), & t\in[0,l/2]\\ \rho_j(l/2), & t\in(l/2,l]\end{cases}$$
$$(i\neq j,\ i,j=1,2).$$

Then the relation

$$\rho_1\rho_2=\tilde{\rho}_1\cdot\tilde{\rho}_2\in \mathrm{P}_0 \text{ və } H_\omega^{\rho_1\rho_2}=H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$$

is satisfied.

The following theorem is held about the representation of Banach spaces of weighted generalized Hölder spaces H_ω^ρ in the form of sums and finding a criterion for the coincidence of the corresponding pairs with each other:

Theorem 2. $H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}=H_\omega^{\tilde{\rho}_1}+H_\omega^{\tilde{\rho}_2}$.

Lets note that this equality is understood in the meaning of space.

Proof. In order to prove the theorem, we should show the satisfaction of the following conditions:

1) $H_\omega^{\tilde{\rho}_1}+H_\omega^{\tilde{\rho}_2}\subset H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$;

2) $H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}\subset H_\omega^{\tilde{\rho}_1}+H_\omega^{\tilde{\rho}_2}$;

3) $\forall u\in H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}\quad \|u\|_{H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}}\approx \inf\limits_{u=u_1+u_2}\left(\|u_1\|_{H_\omega^{\tilde{\rho}_1}}+\|u_2\|_{H_\omega^{\tilde{\rho}_2}}\right)$.

Firstly, lets prove the 1) :

Lets take $u\in H_\omega^{\tilde{\rho}_1}$. Then, because of

$\varphi_0\overset{df}{=}u\tilde{\rho}_1\tilde{\rho}_2=\varphi(x)\tilde{\rho}_2(x),\ \varphi(x)\in H_\omega$; $\varphi(0)=0,\ \tilde{\rho}_2(l)=0$ we get $\varphi_0(0)=\varphi_0(l)=0$ and because of $\omega_{\tilde{\rho}}(\delta)\le c\omega(\delta)$

we get $\varphi_0(0)=H_\omega$, i.e. $u\in H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$.

Thus, $H_\omega^{\tilde{\rho}_1}\subset H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$.

Analogically, we can show that $H_\omega^{\tilde{\rho}_2}\subset H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$.

Taking into account that $H_\omega^{\tilde{\rho}_1\tilde{\rho}_2}$ is the linear set, then we get that

$$H_\omega^{\tilde{\rho}_1}+H_\omega^{\tilde{\rho}_2}\subset H_\omega^{\rho_1\rho_2},$$

i.e. the 1) is proved.

Now, lets prove 2).

Let $u \in H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}$. Then, if we take

$$u_1 = \begin{cases} u(x), & x \in (0, l/2] \\ u(l/2), & x \in [l/2, l] \end{cases}$$

$$u_2 = u - u_1 = \begin{cases} 0, & x \in (0, l/2] \\ u(x) - u(l/2), & x \in [l/2, l] \end{cases}$$

we get $u = u_1 + u_2$ and $u_1 \in H_{\omega}^{\tilde{\rho}_1}$, $u_2 \in H_{\omega}^{\tilde{\rho}_2}$. Thus, $\exists u_1 \in H_{\omega}^{\tilde{\rho}_1}$ for $\forall u \in H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}$, $u_2 \in H_{\omega}^{\tilde{\rho}_2}$: $u = u_1 + u_2 \Rightarrow H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2} \subset H_{\omega}^{\tilde{\rho}_1} + H_{\omega}^{\tilde{\rho}_2}$, i.e. the 2) is proved.

Now, lets prove the 3):

Lets take $u \in H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}$. Pay attention to functions u_1 and u_2 from 2):

$$x, y \in (0, l/2] \text{ for } (x < y)$$

$$|(u\rho)(x) - (u\rho)(y)| = |(u_1\rho)(x) - (u_1\rho)(y) + (u_2\rho)(x) - (u_2\rho)(y)| =$$

$$= |(u_1\rho)(x) - (u_1\rho)(y)| = \rho_1(l/2)|(u_1\tilde{\rho}_1)(x) - (u_1\tilde{\rho}_1)(y)|.$$

Note that, if $x, y \in (0, l/2]$ we get $\rho_1(t)\rho_2(t) = \tilde{\rho}_1(t)\rho_1(l/2)$ and $(u_2\rho)(x) - (u_2\rho)(y) = 0$. Thus, if $0 < x < y < l/2$

$$|(u\rho)(x) - (u\rho)(y)| = \rho_1(l/2)|(u_1\tilde{\rho}_1)(x) - (u_1\tilde{\rho}_1)(y)| \quad (*)$$

is satisfied.

And also, because of $(u_1\tilde{\rho}_1)(\xi) = u(l/2)\tilde{\rho}_1(l/2)$ inequality (*) for $\forall x \in (0, l)$ is held if $\xi \in [l/2, l]$. Then we get that,

$$\|u\|_{H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}} \geq \rho_1(l/2) \cdot \|u_1\|_{H_{\omega}^{\tilde{\rho}_1}}.$$

Analogically, can be proved that

$$\|u\|_{H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}} \geq \rho_2(l/2) \cdot \|u_2\|_{H_{\omega}^{\tilde{\rho}_2}}.$$

Then we get that, $u_1 \in H_{\omega}^{\tilde{\rho}_1}$, $u_2 \in H_{\omega}^{\tilde{\rho}_2}$: $u = u_1 + u_2$ for the function $\forall u \in H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}$ and inequality

$$\|u\|_{H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}} \geq \max\{\rho_1(l/2), \rho_2(l/2)\} \cdot \left(\|u_1\|_{H_{\omega}^{\tilde{\rho}_1}} + \|u_2\|_{H_{\omega}^{\tilde{\rho}_2}}\right) \geq$$

$$\geq \max\{\rho_1(l/2), \rho_2(l/2)\} \cdot \inf_{u = u_1 + u_2} \left(\|u_1\|_{H_{\omega}^{\tilde{\rho}_1}} + \|u_2\|_{H_{\omega}^{\tilde{\rho}_2}}\right) =$$

$$=\max\{\rho_1(l/2),\rho_2(l/2)\}\cdot\|u\|_{H_\omega^{\tilde\rho_1}+H_\omega^{\tilde\rho_2}}$$

is satisfied.

Thus, for $\forall u\in H_\omega^{\tilde\rho_1\tilde\rho_2}$

$$\|u\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\geq c_1\cdot\|u\|_{H_\omega^{\tilde\rho_1}+H_\omega^{\tilde\rho_2}},\ c_1=\max\{\rho_1(l/2),\rho_2(l/2)\}. \quad (1)$$

Now, lets take $u_1\in H_\omega^{\tilde\rho_1}$, $u_2\in H_\omega^{\tilde\rho_2}$ and $u=u_1+u_2$. Then, because of $H_\omega^{\tilde\rho_1}\subset H_\omega^{\tilde\rho_1\tilde\rho_2}$ and $H_\omega^{\tilde\rho_2}\subset H_\omega^{\tilde\rho_1\tilde\rho_2}$ we get that

$$\|u\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\leq\|u_1\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}+\|u_2\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}} \quad (2)$$

Lets prove that $\forall c_1>0\quad u_1\in H_\omega^{\tilde\rho_1}, \|u\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\leq c\cdot\|u_1\|_{H_\omega^{\tilde\rho_1}}$

Certainly, because of $\|u_1\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}=\sup\frac{|(u_1\tilde\rho_1\tilde\rho_2)(x)-(u_1\tilde\rho_1\tilde\rho_2)(y)|}{\omega(|x-y|)}$, define that

$$u_1\tilde\rho_1=\varphi.$$

Taking into account that $|\varphi(x)|\leq\|u\|_{H_\omega^{\tilde\rho_1}}\omega(x)$ və $|\varphi(x)-\varphi(y)|\leq\|u_1\|_{H_\omega^{\tilde\rho_1}}\omega(|x-y|)$,

we get that:

$$|(u_1\tilde\rho_1\tilde\rho_2)(x)-(u_1\tilde\rho_1\tilde\rho_2)(y)|=|\varphi(x)\tilde\rho_2(x)-\varphi(y)\tilde\rho_2(y)|=|(\varphi(x)-\varphi(y))\tilde\rho_2(x)+$$

$$+\varphi(y)(\tilde\rho_2(x)-\tilde\rho_2(y))|\leq\|u_1\|_{H_\omega^{\tilde\rho_1}}\cdot\omega(|x-y|)\max\tilde\rho_2(x)+\|u_1\|_{H_\omega^{\tilde\rho_1}}\omega(y)(\tilde\rho_2(x)-\tilde\rho_2(y))\leq$$

$$\leq\|u_1\|_{H_\omega^{\tilde\rho_1}}\cdot(\omega(|x-y|)+\tilde\rho_i(y)\omega(y))\leq c\cdot\|u_1\|_{H_\omega^{\tilde\rho_1}},$$

here $c=\omega(|x-y|)+\tilde\rho_i(y)\omega(y),\ i=1,2$.

In the same way, we can show that

$$\|u_2\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\leq c\cdot\|u_2\|_{H_\omega^{\tilde\rho_2}}.$$

Then, taking into account equation (2)] $u=u_1+u_2$, $u_1\in H_\omega^{\tilde\rho_1}$, $u_2\in H_\omega^{\tilde\rho_2}$ and

$$\|u_2\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\leq c\cdot(\|u_1\|_{H_\omega^{\tilde\rho_2}}+\|u_2\|_{H_\omega^{\tilde\rho_2}})$$

Is satisfied. I.e. ,

$$\|u\|_{H_\omega^{\tilde\rho_1\tilde\rho_2}}\leq c\cdot\|u\|_{H_\omega^{\tilde\rho_2}+H_\omega^{\tilde\rho_2}} \quad (3)$$

doğrudur.

Thus, from relations (1) və (3) we get that , the relation

$$\|u\|_{H_{\omega}^{\tilde{\rho}_1\tilde{\rho}_2}} \approx \inf_{u=u_1+u_2}\left(\|u_1\|_{H_{\omega}^{\tilde{\rho}_2}} + \|u_2\|_{H_{\omega}^{\tilde{\rho}_2}}\right)$$

is valid.

With that, the theorem 2 is proved.

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