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SPECTRAL ANALYSIS OF THE AIRY OPERATOR WITH A STEP-LIKE POTENTIAL

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Abstract

The Airy operator $L=-\frac{d^2}{dx^2}+k_1x\theta(x)+k_2x\theta(-x)$, where $\theta(x)$ is Heaviside function, $k_1 \neq 0, k_2 \neq 0$ are real numbers, is considered. The scattering problem for the operator L is studied. A formula for expansion in terms of eigenfunctions of a continuous spectrum is obtained.

Keywords: The Airy operator, step-like potential, the Airy function, scattering problem, formula of the expansion.

Mathematics Subject Classification: 34A55, 34B20, 34L05.

1. Introduction

The operator $S=-\frac{d^2}{dx^2}+x$ describes the effect of the potential on the electric field and is called the Airy operator. The spectral properties of the Airy operator has been intensively studied during the many years (see [4], [6],[7]and references quoted therein). It is known (see [2]) that in the study of the inverse scattering problem, a special role is played by expansions in eigenfunctions of the continuous spectrum of the unperturbed operator.

We consider the differential equation

$$-y''+\rho(x)xy=\lambda y,\ -\infty<x<\infty,\quad \lambda\in C, \tag{1}$$

where $\rho(x)=k_1^3\theta(x)+k_2^3\theta(-x)$, $\theta(x)$ is Heaviside function, $k_1\neq 0, k_2\neq 0$ are real numbers. This equation corresponds to the Airy operator $S=-\frac{d^2}{dx^2}+\rho(x)x+q(x)$, the perturbation potential $q(x)$ of which has a step-like form. Differential equation (1) defines in space $L_2(-\infty,+\infty)$ a self-adjoint operator L, which can be obtained by closure of symmetric operator defined by equation (1) on twice continuously differentiable finite functions. In this paper, the direct scattering problem for the operator L is studied. A formula is obtained for the expansion in terms of eigenfunctions of the continuous spectrum of the operator L. The obtained results can be used to solve inverse scattering problem for the Airy operator $S=-\frac{d^2}{dx^2}+\rho(x)x+q(x)$, the perturbation potential $q(x)$ of which satisfy the conditions

$$q(x)\to 0, x\to\pm\infty.$$

Note that various spectral problems for the Airy equation were studied in the works [3], [5]- [9], [11].

2. Spectral analysis of the operator L

In what follows, we deal with special functions satisfying the Airy equation

$$-y''+zy=0.$$

It is well known (e.g., see [1]) that this equation has two linearly independent solutions $Ai(z)$ and $Bi(z)$ with the initial conditions

$$Ai(0)=\frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, Ai'(0)=\frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$

$$Bi(0)=\frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}, Bi'(0)=\frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

The Wronskian $\{Ai(z), Bi(z)\}$ of these functions satisfies

$$\{Ai(z), Bi(z)\}=Ai(z)Bi'(z)-Ai'(z)Bi(z)=\pi^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. We have (see [1]) asymptotic equalities for $|z|\to\infty$

$$\mathrm{A}i(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right], |\arg z| < \pi$$

$$\mathrm{A}i(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{2\pi}{3},$$

$$Bi(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\left[1+O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{\pi}{3},$$

$$Bi(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \cos\left(\zeta+\frac{\pi}{4}\right)\left[1+O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{2\pi}{3}.$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$.

Lemma2.1. *For any λ from the complex plane, equation (1) has solutions $\psi_{\pm}(x,\lambda)$ in the form*

$$\psi_{+}(x,\lambda) = \begin{cases} Ai(k_1 x-\lambda), x \ge 0, \\ \left[1-\pi\left(\frac{k_1}{k_2}-1\right)Ai(-\lambda)Bi'(-\lambda)\right]Ai(k_2 x-\lambda)+ \\ +\pi\left(\frac{k_1}{k_2}-1\right)Ai(-\lambda)Ai'(-\lambda)Bi(k_2 x-\lambda), x<0, \end{cases} \tag{2}$$

$$\psi_{-}(x,\lambda) = \begin{cases} \left[1+\pi\left(1-\frac{k_2}{k_1}\right)Ai'(-\lambda)Bi(-\lambda)\right]Ai(k_1 x-\lambda)+ \\ +\frac{k_2}{k_1}Ai(-\lambda)Ai'(-\lambda)Bi(k_1 x-\lambda), x \ge 0, \\ Ai(k_2 x-\lambda)-iBi(k_2 x-\lambda), x<0. \end{cases} \tag{3}$$

Proof. Obviously, when $x \ge 0$ one of the solutions of equation (4) is function $Ai(k_1 x-\lambda)$. On the other hand, for $x \le 0$ functions $Ai(k_2 x-\lambda)$ and $Bi(k_2 x-\lambda)$ form a fundamental system of solutions to equation (1). Therefore, any solution of equation (1) can be represented as

$$\alpha Ai(k_2 x-\lambda)+\beta Bi(k_2 x-\lambda). \tag{4}$$

If we glue these solutions at a point $x=0$, we get

$$\begin{cases} Ai(-\lambda)\alpha + Bi(-\lambda)\beta = Ai(-\lambda), \\ Ai'(-\lambda)\alpha + Bi'(-\lambda)\beta = \dfrac{k_1}{k_2} Ai'(-\lambda). \end{cases}$$

Using Cramer's rule, from the last system of equations we obtain

$$\alpha = \pi \begin{vmatrix} Ai(-\lambda) & Bi(-\lambda) \\ \dfrac{k_1}{k_2} Ai'(-\lambda) & Bi'(-\lambda) \end{vmatrix}, \beta = \pi \begin{vmatrix} Ai(-\lambda) & Ai(-\lambda) \\ Ai'(-\lambda) & \dfrac{k_1}{k_2} Ai'(-\lambda) \end{vmatrix}.$$

Substituting the found values of α and β into representation (4), we obtain formula (2). Formula (3) is derived similarly.

The lemma is proved.

We note that at each fixed x, the solutions $\psi_{\pm}(x,\lambda)$ are the entire functions with respect to λ. Moreover, the solution $\psi_{+}(x,\lambda)$ is real-valued for $\lambda \in (-\infty,+\infty)$.

Next, using (2) and (3), we find that for $\lambda \in (-\infty,+\infty)$ two solutions $\psi_{-}(x,\lambda)$, $\overline{\psi_{-}(x,\lambda)}$ of Eq. (4) are linearly independent and their Wronskian is given by

$$\{\psi_{-}(x,\lambda), \overline{\psi_{-}(x,\lambda)}\} = \psi_{-}(0,\lambda)\overline{\psi'_{-}(0,\lambda)} - \psi'_{-}(0,\lambda)\overline{\psi_{-}(0,\lambda)} = 2ik_2\pi^{-1} .$$

It follows from the last equality that the identity

$$\psi_{+}(x,\lambda) = a_0(\lambda)\overline{\psi_{-}(x,\lambda)} + \overline{a_0(\lambda)}\psi_{-}(x,\lambda), \tag{5}$$

holds for $\lambda \in (-\infty,+\infty)$, where the coefficient $a_0(\lambda)$, by virtue of (2), (3), is given by

$$a_0(\lambda) = \frac{\pi W\{\psi_{-}(x,\lambda), \psi_{+}(x,\lambda)\}}{2ik_2} =$$

$$= \frac{\pi}{2ik_2}\left([Ai(-\lambda) - iBi(-\lambda)]k_1 Ai'(-\lambda) - [Ai'(-\lambda) - iBi'(-\lambda)]k_2 Ai(-\lambda)\right). \tag{6}$$

According to formula (6), the function $a_0(\lambda)$ admits an analytic extension to the all complex plane and has no zeros. The functions $t_0(\lambda) = \dfrac{1}{a_0(\lambda)}$ and $r_0(\lambda) = \dfrac{\overline{a_0(\lambda)}}{a_0(\lambda)}$ have the meaning of the respective transition and reflection coefficients in the scattering theory for the equation (1).

The function $\frac{\psi_+(x,\lambda)}{a_0(\lambda)}$ is called the solution of the scattering problem for the equation (3). For real λ, the solution $\frac{\psi_+(x,\lambda)}{a_0(\lambda)}$ is bounded, which corresponds to the continuous spectrum of problem (1).

Let us study the resolvent of the operator L. We consider the equation

$$-y'' + \rho(x)xy - \lambda y = f(x), \ -\infty < x < \infty, \quad \operatorname{Im}\lambda \neq 0,$$

where $y = y(x)$, $f(x) \in L_2(-\infty,+\infty)$. By a classical theorem on the general form of a solution of a differential equation,

$$y(x) = D_+\psi_+(x,\lambda) + D_-\psi_-(x,\lambda) +$$
$$+ \frac{\pi i}{2k_2 a_0(\lambda)}\left[\psi_+(x,\lambda)\int_{-\infty}^{x}\psi_-(t,\lambda)f(t)dt + \psi_-(x,\lambda)\int_{x}^{+\infty}\psi_+(t,\lambda)f(t)dt\right],$$

where D_+ and D_- are constants. From formulas (2), (3) it follows that

$$\psi_-(x,\lambda) \in L_2(-\infty,0), \ \psi_+(x,\lambda) \notin L_2(-\infty,0),$$
$$\psi_+(x,\lambda)\int_{-\infty}^{x}\psi_-(t,\lambda)f(t)dt + \psi_-(x,\lambda)\int_{x}^{+\infty}\psi_+(t,\lambda)f(t)dt \in L_2(-\infty,+\infty).$$

Then from relations $y(x) \in L_2(-\infty,+\infty)$, $\psi_-(x,\lambda) \notin L_2(0,+\infty)$, $\psi_+(x,\lambda) \notin L_2(-\infty,0)$, $\psi_+(x,\lambda) \in L_2(0,+\infty)$, $\psi_-(x,\lambda) \in L_2(-\infty,0)$ it follows that $D_+ = 0$, $D_- = 0$. Thus, formula

$$y(x) = \frac{\pi i}{2k_2 a_0(\lambda)}\left[\psi_+(x,\lambda)\int_{-\infty}^{x}\psi_-(t,\lambda)f(t)dt + \psi_-(x,\lambda)\int_{x}^{+\infty}\psi_+(t,\lambda)f(t)dt\right],$$

defines the inverse operator $(L - \lambda I)^{-1}$, where I is the unit operator. It is easy to prove that the inverse operator $(L - \lambda I)^{-1}$ is bounded.

Thus, we have proven the following theorem.

Theorem 2. 1. *For $\lambda \notin (-\infty,+\infty)$, integral operator R_λ is defined in space $L_2(-\infty,+\infty)$ by the formula*

$$(R_\lambda f)(x) = \int_{-\infty}^{+\infty} R(x,t,\lambda)f(t)dt,$$

where

$R(x,t,\lambda)=\frac{\pi i}{2k_2a_0(\lambda)}\psi_+(x,\lambda)\psi_-(x,\lambda)\theta(x-t)+\psi_-(x,\lambda)\psi_+(t,\lambda)\theta(t-x)$ *is the resolvent of the operator* L.

Explicit formula for the resolvent R_λ leads to the theorem of expansion in terms of eigenfunctions of the operator L. As is known (see [4]), the continuous spectrum of the operator L fills the entire real axis. Then, we denote by $E(\Delta)$, where Δ runs the Borel subsets in $(-\infty,+\infty)$, decomposition of the identity of a self-adjoint operator L (see. [10]). In the absence of a point spectrum, the following formula is valid:

$$E(\Delta)=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\Delta\left(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\right)d\lambda.$$

(see [10]). This formula is sometimes called Stone's formula. In particular, assuming $\Delta=(-\infty,+\infty)$, for the operator L we get

$$I=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\right)d\lambda$$

This formula and relation (5) serve as the basis for the derivation of the expansion theorem.

Theorem 2. 2. *The expansion formula*

$$\frac{1}{4}\int_{-\infty}^{\infty}\frac{1}{\left|k_2a_0(\lambda)\right|^2}\psi_+(x,\lambda)\psi_+(y,\lambda)d\lambda=\delta(x-y)$$

is valid, where δ *is Dirac's delta function.*

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