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Global bifurcation from infinity in nonlinearizable Dirac problems with a spectral parameter in the boundary conditions

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Abstract

In this paper we consider global bifurcation from infinity in nonlinearizable Dirac problem with a spectral parameter contained in both boundary conditions. We prove the existence of two families of unbounded components of the set of nontrivial solutions to this problem, which bifurcate from asymptotic intervals and contained in classes of vector-functions possessing oscillatory properties of the eigenvector-functions of the corresponding linear Dirac problem in the neighborhood of these intervals.

Keywords: nonlinearizable Dirac problem, eigenvalue, eigenvector-function, bifurcation interval, unbounded component
Mathematics Subject Classification *(2020): 34A30, 34B15, 34C10, 34K29, 47J10, 47J15*

1. Introduction

In this paper we consider the following nonlinear Dirac problem

$$Bw'(x) - P(x)w(x) = \lambda w(x) + f(x, w(x), \lambda) + g(x, w(x), \lambda),\ x \in (0, \pi), \tag{1}$$

$$U_1(\lambda, w) = (\lambda \cos\alpha + a_0, \lambda \sin\alpha + b_0) w(0) = 0, \tag{2}$$

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$$U_2(\lambda, w) = (\lambda \cos\beta + a_1, \lambda \sin\beta + b_1) w(\pi) = 0, \qquad (3)$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}, \ w(x) = \begin{pmatrix} u(x) \\ \vartheta(x) \end{pmatrix},$$

$\lambda \in R$ is an eigenvalue parameter, $p, r \in C([0,\pi]; R)$, $a_0, b_0, a_1, b_1, \alpha$ and β are real constants such that

$$0 \le \alpha, \beta < \pi$$

and

$$\sigma_0 = a_0 \sin\alpha - b_0 \cos\alpha < 0, \ \sigma_1 = a_1 \sin\beta - b_1 \cos\beta > 0. \qquad (4)$$

Here the real valued-functions functions $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ are continuous on $[0,\pi] \times R^3$ and satisfy the following conditions:

$$| f_1(x, w, \lambda) | \le K | w |, \ | f_2(x, w, \lambda) | \le L | w |, \ (x, w, \lambda) \in [0,\pi] \times R^3, \qquad (5)$$

where K and L are some positive constants;

$$g(x, w, \lambda) = o(| w |) \text{ as } | w | \to \infty, \qquad (6)$$

uniformly in $(x, \lambda) \in [0,\pi] \times \Lambda$, for any bounded interval $\Lambda \subset R$.

The Dirac equation, which is a relativistic wave equation for describing spin-$1/2$ particles, i.e., fermions, underlies the formulation of relativistic quantum mechanics. This equation has wide applications in the physical sciences, ranging from high energy physics, quantum information and quantum electrodynamics [21].

Nonlinear Dirac equations are widely used in various fields of physics, including atomic, nuclear and gravitational physics. These equations describe the behaviour of fermions in the presence of external electromagnetic fields, modelled by an electric and magnetic potential and taking into account the nonlinear self-interaction of particles. In addition, they provide invaluable information about the behaviour of matter under extreme conditions (see [12, 14-16, 20-22].

The global bifurcation from zero and infinity of nontrivial solutions to nonlinear Sturm-Liouville problems of second and fourth order was studied in [3-6, 8, 11, 17-19]. In these papers it was shown that there are global components of the sets of nontrivial solutions to these problems bifurcating from points and intervals of the lines $R \times \{0\}$ and $R \times \{\infty\}$, and contained in classes of functions with fixed oscillation count in the neighbourhood of these bifurcation points and intervals. Similar results for one-dimensional nonlinear Dirac systems were obtained in [7, 9, 13] in the case when the boundary conditions do not depend on the spectral parameter, and in [10] in the case when one of the boundary conditions depends on the spectral parameter.

Note that problem (1)-(3) in the case $f \equiv 0$ was studied in [1], and in the case when g satisfies condition (6) in the neighbourhood of zero, it was studied in [2]. In these papers, the authors established results similar to those stated above.

The purpose of this work is to study the structure of bifurcation points, the structure and behaviour of the global components of the set of nontrivial solutions to problem (1)-(3) under conditions (4)-(6).

2. Preliminary

Let $E = C([0,\pi]; R^2)$ be the Banach space with the norm $\| w \| = \| u \|_\infty + \| \vartheta \|_\infty$, where $\| u \|_\infty = \max_{x \in [0,\pi]} | u(x) |$.

We define S be the subset of E given by

$$S = \{ w \in E \,|\, | u(x) | + | \vartheta(x) | > 0, \; x \in [0,\pi] \}.$$

Let $\gamma(\lambda)$ and $\delta(\lambda)$ be continuous functions on R such that

$$\cot \gamma(\lambda) = -\frac{\lambda \cos \alpha + a_0}{\lambda \sin \alpha + b_0}, \quad \gamma\left(-\frac{b_0}{\sin \alpha} \right) = 0 \text{ for } \alpha \neq 0,$$

$$\cot \delta(\lambda) = -\frac{\lambda \cos \beta + a_1}{\lambda \sin \beta + b_1}, \quad \delta\left(-\frac{b_0}{\sin \alpha} \right) = 0 \text{ for } \beta \neq 0.$$

By [1, (2.3), (2.4)] we have

$$\gamma(\lambda) \in (-\alpha, \pi - \alpha), \; \delta(\lambda) \in (-\beta, \pi - \beta). \tag{7}$$

For each $\lambda \in R$ and each $w \in E,$ we define the function $\theta(w, \lambda, x)$ continuous on $[0,\pi]$ by

$$\cot \theta(\lambda, w, x) = \frac{u(x)}{\vartheta(x)}, \; \theta(\lambda, w, 0) = \gamma(\lambda). \tag{8}$$

We consider the linear spectral parameter

$$\begin{cases} \ell(w)(x) = \lambda w(x), \; x \in (0,\pi), \\ U(\lambda, w) = \tilde{0}, \end{cases} \tag{9}$$

where

$$U(\lambda, w) = \begin{pmatrix} U_1(\lambda, w) \\ U_2(\lambda, w) \end{pmatrix}, \ \tilde{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By [1, Remark 2.1] that the eigenvalues $\lambda_k, k \in Z,$ of problem (9) are real, simple and can be numbered in ascending order on the real axis as follows

$$. . . < \lambda_{-k} < . . . < \lambda_{-1} < \lambda_0 < \lambda_1 < . . . < \lambda_k < . . . .$$

Moreover, if for each $k \in Z$ we denote by w_k the eigenvector-function corresponding to the eigenvalue λ_k, then the angular function $\theta(\lambda_k, w_k, x)$ at $x = 0$ and $x = \pi$ will satisfy the following relations (see [1, (2.7)])

$$\theta(\lambda_k, w_k, 0) = \gamma(\lambda_k) \ \text{ and } \ \theta(\lambda_k, w_k, x) = \delta(\lambda_k) + k\pi. \tag{10}$$

We define the integers m_1 and m_{-1} as follows:

$$\begin{aligned} m_1 &= \min \{ k \in \mathrm{Z} \mid \lambda_k + p(x) > 0, \lambda_k + r(x) > 0, x \in [0, \pi] \}, \\ m_{-1} &= \max \{ k \in \mathrm{Z} \mid \lambda_k + p(x) < 0, \lambda_k + r(x) < 0, x \in [0, \pi] \} \end{aligned} \tag{11}$$

Remark 2.1. By (11), the first and second parts of statement (ii) of [10, Theorem 2.4] hold for $\theta(\lambda_k, w_k, x)$ for all $k \ge m_1$ and $k \le m_{-1}$, respectively.

For each $k \in \mathrm{Z},\ k \le m_{-1}$ or $k \ge m_1$ and each $\lambda \in R$ by $S_{k,\lambda}^+$ we denote the set of vector-functions $w \in S$ such that (see [1, Section 2])

(i) $\theta(\lambda, w, \pi) = \delta(\lambda)$;

(ii) if $k \ge m_1$, then for fixed λ and w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k \le m_{-1}$, then for fixed λ and w, as x increases the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above;

(iii) the function $u(x)$ is positive in a deleted neighborhood of $x = 0$.

Let $S_{k,\lambda}^- = S_{k,\lambda}^+$ and $S_{k,\lambda} = S_{k,\lambda}^+ \bigcup S_{k,\lambda}^+$. For each $\lambda \in R$ the sets $S_{k,\lambda}^+$, $S_{k,\lambda}^+$ and $S_{k,\lambda}$ for $k \le m_{-1}$ and $k \ge m_1$ are disjoint and open in E. Moreover, if $w \in \partial S_{k,\lambda}^+$ or $w \in \partial S_{k,\lambda}^-$, then there exists $\xi \in [0, \pi]$ such that $| w(\xi) | = 0$ ([10, Remark 2.7]).

For each $k \in \mathrm{Z},\ k \le m_{-1}$ or $k \ge m_1$ we introduce the following sets

$$S_k^+ = \bigcup_{\lambda \in R} S_{k,\lambda}^+, \ \ S_k^- = -S_k^+ \ \text{ and } \ S_k = S_k^+ \bigcup S_k^-.$$

Let $\hat{E}$ be the Banach space $E \oplus R^2$ with the norm given by

$$\| \hat{w} \|_0 = \| (w, s, t)^t \|_0 = \| w \| + | s | + | t |,$$

and

$$\hat{S} = \{\hat{w} \in E \mid w \in S\}.$$

For each $k \in Z,\ k \le m_{-1}$ or $k \ge m_1,$ and each ν by $\hat{S}_k^+,\ \hat{S}_k^-$ and $\hat{S}_k$ we denote the subsets of $\hat{S}$ such that

$$\hat{S}_k^+ = \{\hat{w} \in \hat{S} \mid w \in S_k^+\},\ \ \hat{S}_k^- = \{\hat{w} \in \hat{S} \mid w \in S_k^-\} \text{ and } \hat{S}_k = \{\hat{w} \in S \mid w \in S_k\}.$$

It is obvious that these sets are disjoint and open in $\hat{E},$ and if $\hat{w} \in \partial S_k$, then there exists $\xi \in [0, \pi]$ such that $|w(\xi)| = 0.$

Let A be the operator on $\hat{E}$ defined by

$$A\hat{w} = A\begin{pmatrix} w \\ s \\ t \end{pmatrix} = \begin{pmatrix} \ell(w) \\ a_0\vartheta(0) + b_0 u(0) \\ a_1\vartheta(\pi) + b_1 u(\pi) \end{pmatrix},$$

$$D(A) = \{\hat{w} \in \hat{E} \mid w \in C^1([0,\pi]; R^2),\ s = -(\vartheta(0)\cos\alpha + u(0)\sin\alpha),$$
$$t = -(\vartheta(\pi)\cos\beta + u(\pi)\sin\beta)\},$$

where the norm of the space $C^1([0,\pi]; R^2)$ is given as

$$\|\hat{w}\|_1 = \|(w,s,t)^t\|_0 = \|w\| + \|w'\| + |s| + |t|.$$

Then problem (9) takes the following equivalent form

$$Aw = \lambda w,\ w \in D(A). \qquad (12)$$

Note that A is a closed operator with a compact resolvent.

As norms in $R \times E$ and $R \times \hat{E}$, we take

$$\|(\lambda, w)\| = \{|\lambda|^2 + \|w\|^2\}^{\frac{1}{2}} \text{ and } \|(\lambda, \hat{w})\|_0 = \{|\lambda|^2 + \|\hat{w}\|_0^2\}^{\frac{1}{2}},$$

respectively.

Now let the nonlinear operators $F : R \times \hat{E} \to \hat{E}$ and $G : R \times \hat{E} \to \hat{E}$ are defined by

$$F(\lambda, w) = F\left(\lambda, \begin{pmatrix} w \\ s \\ t \end{pmatrix}\right) = \begin{pmatrix} f(x, w, \lambda) \\ 0 \\ 0 \end{pmatrix},$$

$$G(\lambda, w) = G\left(\lambda, \begin{pmatrix} w \\ s \\ t \end{pmatrix}\right) = \begin{pmatrix} g(x, w, \lambda) \\ 0 \\ 0 \end{pmatrix}$$

Then problem (1)-(3) is equivalent to the following operator equation

$$A\hat{w} = \lambda\hat{w} + F(\lambda, \hat{w}) + G(\lambda, \hat{w}), \tag{13}$$

i.e., between the solutions of problems (1)-(3) and (13) there is the following one-to-one correspondence

$$(\lambda, w) \leftrightarrow (\lambda, \hat{w}) = (\lambda, (w,\, s,\, t)^t),\;\; s = -(\vartheta(0)\cos\alpha + u(0)\sin\alpha), \\ t = -(\vartheta(\pi)\cos\beta + u(\pi)\sin\beta). \tag{14}$$

If $f \equiv 0$, then problem (13) takes the following form

$$A\hat{w} = \lambda\hat{w} + G(\lambda, \hat{w}). \tag{15}$$

Problem (14) (or (1)-(3) with $f \equiv 0$) was investigated in [1], where the following result was proved.

Theorem 1 [1, Theorem 3.1]. *For each $k \in Z$, $k \le m_{-1}$ or $k \ge m_1$, and each $\nu \in \{+,-\}$ there exists a component $\hat{C}_k^{\nu}$ of the set of nontrivial solutions of problem (15) that meet (λ_k, ∞) with respect to the set $R \times \hat{S}_k^{\nu}$ and for this set at least one of the following statements holds:*

(i) $\hat{C}_k^{\nu}$ *meets* (λ'_k, ∞) *with respect to the set* $R \times \hat{S}_{k'}^{\nu'}$ *for some* $(k', \nu') \ne (k, \nu)$;

(ii) $\hat{C}_k^{\nu}$ *meets* $R \times \{\tilde{0}\}$ *for some* $\lambda \in R$;

(iii) *the natural projection* $P_{R\times\{\tilde{0}\}}(\hat{C}_k^{\nu})$ *of* $\hat{C}_k^{\nu}$ *onto* $R \times \{\tilde{0}\}$ *is unbounded.*

3. Global bifurcation from infinity in problem (1)-(3)

In this section we consider global bifurcation of nontrivial solutions to problem (13) in the case when the function f is not identically zero.

We define the continuous operators $\tilde{F} : R \times \hat{E} \to \hat{E}$ and $\tilde{G} : R \times \hat{E} \to \hat{E}$ as follows:

$$\tilde{F}(\lambda, \hat{w}) = \begin{cases} \|\hat{w}\|_0^2 \, F\left(\lambda, \dfrac{\hat{w}}{\|\hat{w}\|_0^2}\right) & if \;\; \hat{w} \ne \hat{0}, \\ \hat{0} & if \;\; \hat{w} = \hat{0}, \end{cases}$$

$$\tilde{G}(\lambda, \hat{w}) = \begin{cases} \|\hat{w}\|_0^2 \, G\left(\lambda, \dfrac{\hat{w}}{\|\hat{w}\|_0^2}\right) & if \;\; \hat{w} \ne \hat{0}, \\ \hat{0} & if \;\; \hat{w} = \hat{0}, \end{cases}$$

Then it follows from Step 2 of the proof of [1, Theorem 3.1] that the operator $\tilde{G}$ is completely continuous. Moreover, by (6) it follows from [9, Lemma 2] that for any sufficiently small $\varepsilon > 0$ there exists a sufficiently large $\rho_\varepsilon > 0$ such that

$$|g(x,w,\lambda)|<\varepsilon|w| \text{ for any } x\in[0,\pi],\ w\in E,\ \|w\|>\rho_\varepsilon,\ \lambda\in\Lambda, \tag{16}$$

where $\Lambda\subset R$ is any bounded interval. By [1, relation (3.3)] for any $\hat{w}\in\hat{E}$ we have

$$\|\hat{w}\|_0\le 3\|w\|. \tag{17}$$

Then, choose

$$\hat{\rho}_\varepsilon=3\rho_\varepsilon \text{ and } \|\hat{w}\|_0>\hat{\rho}_\varepsilon,$$

we get

$$\|w\|>\rho_\varepsilon.$$

Consequently, it follows from (16) that

$$|g(x,w,\lambda)|<\varepsilon\|w\| \text{ for any } x\in[0,\pi],\ \hat{w}\in E,\ \|\hat{w}\|>\hat{\rho}_\varepsilon,\lambda\in\Lambda. \tag{18}$$

Therefore, we obtain the following relation

$$\frac{\|G(\lambda,\hat{w})\|_0}{\|\hat{w}\|_0}\le\frac{\varepsilon\|w\|}{\|w\|_0}\le\varepsilon,$$

which implies that

$$\|G(\lambda,\hat{w})\|_0=o(\|\hat{w}\|_0) \text{ as } \|\hat{w}\|_0\to\infty, \tag{19}$$

uniformly for $\lambda\in\Lambda$.

By (5) we have

$$\|\widetilde{F}(\lambda,\hat{w})\|_0=\|\hat{w}\|_0^2\left\|F\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_0^2}\right)\right\|_0\le\{K+L\}\|\hat{w}\|_0. \tag{20}$$

It is obvious that if $\|\hat{w}\|_0\to 0$, then $\left\|\frac{\hat{w}}{\|\hat{w}\|_0^2}\right\|_0=\frac{1}{\|\hat{w}\|_0}\to\infty$, and consequently, by (19) we get

$$\frac{\|\widetilde{G}(\lambda,\hat{w})\|_0}{\|w\|_0}=\frac{\|\hat{w}\|_0^2\,G\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_0^2}\right)}{\|w\|_0}=\frac{G\left(\lambda,\frac{\hat{w}}{\|\hat{w}\|_0^2}\right)}{\left\|\frac{w}{\|\hat{w}\|_0^2}\right\|_0}\to 0 \text{ as } \|\hat{w}\|_0\to 0, \tag{21}$$

uniformly for $\lambda\in\Lambda$.

Let $(\lambda,\hat{w})\in D$. Setting $\tilde{w}=\frac{\hat{w}}{\|\hat{w}\|_0^2}$ and dividing (13) by $\|\hat{w}\|_0^2$ yields

$$A\tilde{w}=\lambda\tilde{w}+\widetilde{F}(\lambda,\tilde{w})+\widetilde{G}(\lambda,\tilde{w}), \tag{22}$$

in view of relations $\|\tilde{w}\|_0 = \dfrac{1}{\|\hat{w}\|_0}$ and $\hat{w} = \dfrac{\tilde{w}}{\|\tilde{w}\|_0^2}$. (20) and (21) show that the transformation

$$T:(\lambda,\hat{w}) \to (\lambda,\hat{\tilde{w}}) = \left(\lambda, \frac{\hat{w}}{\|\hat{w}\|_0^2}\right)$$

used earlier in papers [8], [18] and [19] turns a ''bifurcation at infinity'' problem (13) into a ''bifurcation from zero'' problem (22).

Remark 1. By [2, Theorem 3.3] for each $k \in Z,\ k \ge m_1$ or $k \le m_{-1}$, and each $\nu \in \{+,-\}$ the set of bifurcation points to the nonlinear eigenvalue problem (22) with respect to the set $R \times \hat{S}_k^\nu$ is nonempty. But using relation (20), we cannot more accurately describe the location of the bifurcation points of problem (22) with respect to the set $R \times \hat{S}_k^\nu$. Below, using the original form of problem (22) and taking into account (14), we will clarify the bifurcation intervals.

We denote by $\hat{D} \subset R \times \hat{E}$ and $\hat{\tilde{D}} \subset R \times \hat{E}$ the sets of nontrivial solutions to problems (13) and (22), respectively, and let

$$I_k = [\lambda_k - (K + L + 2 + c_k), \lambda_k + (K + L + 2 + c_k)],$$

where $c_k = O(1/k)$.

Lemma 1. *Let* $(\hat{\lambda},\hat{0})$ *be a bifurcation point of problem (22) with respect to the set* $R \times \hat{S}_k^\nu$, $k \in Z,\ k \ge m_1$ *or* $k \le m_{-1}$, *and* $\nu \in \{+,-\}$. *Then* $\hat{\lambda} \in I_k$.

Proof. Note that problem (22) reduces to the following equivalent problem

$$\begin{cases} \ell(w) = \lambda w + \|\hat{\tilde{w}}\|_0^2\ f\left(x, \dfrac{\tilde{w}}{\|\hat{\tilde{w}}\|_0^2}, \lambda\right) + \|\hat{\tilde{w}}\|_0^2\ g\left(x, \dfrac{\tilde{w}}{\|\hat{\tilde{w}}\|_0^2}, \lambda\right),\ x \in (0,\pi), \\ U(\lambda,\tilde{w}) = \tilde{0}. \end{cases} \tag{23}$$

We introduce the following notations

$$\tilde{f}(x,w,\lambda) = \begin{pmatrix} \|\hat{w}\|_0^2\ f_1\left(x, \dfrac{w}{\|\hat{w}\|_0^2}, \lambda\right) \\ \|\hat{w}\|_0^2\ f_2\left(x, \dfrac{w}{\|\hat{w}\|_0^2}, \lambda\right) \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(x,w,\lambda) \\ \tilde{f}_2(x,w,\lambda) \end{pmatrix}, \tag{24}$$

$$\tilde{g}(x,w,\lambda)=\begin{pmatrix} \|\hat{w}\|_0^2 \; g_1\left(x,\dfrac{w}{\|\hat{w}\|_0^2},\lambda\right) \\ \|\hat{w}\|_0^2 \; g_2\left(x,\dfrac{w}{\|\hat{w}\|_0^2},\lambda\right) \end{pmatrix}=\begin{pmatrix} \tilde{g}_1(x,w,\lambda) \\ \tilde{g}_2(x,w,\lambda) \end{pmatrix}. \tag{25}$$

In view of (24), by (5) we obtain

$$\begin{aligned} |\tilde{f}_1(x,w(x),\lambda)| &= \|\hat{w}\|_0^2 \left| f_1\left(x,\frac{w(x)}{\|\hat{w}\|_0^2},\lambda\right)\right| \le K\,|w(x)|, \\ |\tilde{f}_2(x,w(x),\lambda)| &= \|\hat{w}\|_0^2 \left| f_2\left(x,\frac{w(x)}{\|\hat{w}\|_0^2},\lambda\right)\right| \le L\,|w(x)|. \end{aligned} \tag{26}$$

If $\|w\|\to 0,$ then it follows from (17) that $\|\hat{w}\|_0 \to 0.$ Again due to (17) we get

$$\|w\| \ge \frac{1}{3}\|\hat{w}\|_0 .$$

Hence we have

$$\left\|\frac{w}{\|\hat{w}\|_0^2}\right\| = \frac{\|w\|}{\|\hat{w}\|_0^2} \ge \frac{1}{3\|\hat{w}\|_0},$$

whence implies that

$$\left\|\frac{w}{\|\hat{w}\|_0^2}\right\| \to \infty \text{ as } \|w\|\to 0.$$

Then by (6) from (25) we obtain

$$\frac{\|\tilde{g}(x,w,\lambda)\|}{\|w\|} = \frac{\|\hat{w}\|_0^2 \left\| g\left(x,\dfrac{w}{\|\hat{w}\|_0^2},\lambda\right)\right\|}{\|w\|} = \frac{\left\| g\left(x,\dfrac{w}{\|\hat{w}\|_0^2},\lambda\right)\right\|}{\left\|\dfrac{w}{\|\hat{w}\|_0^2}\right\|} \to 0 \text{ as } \|w\|\to 0, \tag{27}$$

uniformly in $\lambda \in \Lambda$.

By correspondence (14) and conditions (26) and (27) it follows from Corollary 3.1 of [2] that for each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$ the bifurcation points of problem (22) with respect to the set $R \times \hat{S}_k^\nu$ is contained in the interval $I_k \times \{\hat{0}\}$. The proof of Lemma 1 is complete.

Applying the transformation T from Remark 1 and Lemma 1, we obtain the following result.

Lemma 2. *For each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$ the set of asymptotic bifurcation points of problem (13) with respect to the set $R \times \hat{S}_k^\nu$ is nonempty. Moreover, if $(\hat{\lambda}, \infty)$ is such a bifurcation point, then $\hat{\lambda} \in I_k$.*

By correspondence (14) Lemma 2 implies the following result.

Lemma 3. *For each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$ the set of asymptotic bifurcation points of problem (1)-(3) with respect to the set $R \times S_k^\nu$ is nonempty. Moreover, if (λ, ∞) is such a bifurcation point, then $\lambda \in I_k$.*

We add points at infinity $(\lambda, \infty), \lambda \in R$, to $R \times E$ and $R \times \hat{E}$, and define the corresponding topologies in the result sets.

For each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$, let $\hat{D}_k^\nu$ be the union of all the components of the set $\hat{D}$ which meet $I_k \times \{\infty\}$ with respect to the set $R \times \hat{S}_k^\nu$. The set $\hat{D}_k^\nu$ may not be connected in $R \times \hat{E}$, but the set $\hat{D}_k^\nu \bigcup (I_k \times \{\infty\})$ is connected in $R \times \hat{E}$.

The main result of this paper is the following theorem.

Theorem 2. *For each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$ the set $\hat{D}_k^\nu$ is nonempty and for this set one of the following assertions hold:*

(i) *$\hat{D}_k^\nu$ meets $I_{k'} \times \{\infty\}$ with respect to the set $R \times \hat{S}_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;*

(ii) *$\hat{D}_k^\nu$ meets $R \times \{\hat{0}\}$ for some $\lambda \in J_k \subset R$;*

(iii) *the natural projection $P_{R \times \{\hat{0}\}}(\hat{D}_k^\nu)$ of $\hat{D}_k^\nu$ onto $R \times \{\hat{0}\}$ is unbounded.*

Proof. For each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$, by $\hat{\tilde{D}}_k^\nu$ we denote the union of all the components of the set $\hat{\tilde{D}}$ which meet $I_k \times \{\hat{0}\}$ with respect to the set $R \times \hat{S}_k^\nu$.

Note that the proof of [1, Theorem 3.1] (see also [10, Theorems 4.5 and 4.6]) is similar to that of [3, Theorem 1.3] with the use very important Lemma 2.8 of [10].

But in our case, for problem (13), this lemma is not applicable outside the neighbourhood of bifurcation intervals since condition (1.6) from [10] for the function g is not satisfied. Therefore, applying the method of proof of Theorem 1.3 of [3] to problem (13) we get the following result: for each $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and each $\nu \in \{+,-\}$, the set $\hat{D}_k^\nu$ is nonempty and either (a) $\hat{\tilde{D}}_k^\nu$ meets $I_{k'} \times \{\hat{0}\}$ with respect to the set $R \times \hat{S}_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$; (b) $\hat{\tilde{D}}_k^\nu$ is unbounded in $R \times \hat{E}$, and two cases are possible: (b_1) the projection $pr(\hat{\tilde{D}}_k^\nu)$ of $\hat{\tilde{D}}_k^\nu$ onto $R \times \{\hat{0}\}$ is bounded and (b_2) this projection is unbounded. It should be noted that in case (b_2) $\hat{\tilde{D}}_k^\nu$ meets some interval $J_k \times \{\infty\}$.

It is obvious that the set $\hat{D}_k^\nu$ is the inverse image $T^{-1}\left(\hat{\tilde{D}}_k^\nu\right)$ of the set $\hat{\tilde{D}}_k^\nu$ under the transformation T. Thus the statements of this theorem follows from the above properties of the set $\hat{\tilde{D}}_k^\nu$ using the transformation T. The proof of this theorem is complete.

Let

$$D = \{ w \in E \mid \hat{w} \in \hat{D} \}$$

and

$$D_k^\nu = \{ w \in E \mid \hat{w} \in \hat{D}_k^\nu \},\ k \in Z,\ k \geq m_1 \textit{ or } k \leq m_{-1}, \textit{ and } \nu \in \{+,-\}.$$

Note that D_k^ν, $k \in Z,\ k \geq m_1$ or $k \leq m_{-1}$, and $\nu \in \{+,-\}$, is the union of all the components of the set D which meet $I_k \times \{\infty\}$ with respect to the set $R \times S_k^\nu$.

According to relation (14), Theorem 2 gives the following result.

Theorem 3. *For each* $k \in Z,\ k \geq m_1$ *or* $k \leq m_{-1}$, *and each* $\nu \in \{+,-\}$ *for the set* D_k^ν *one of the following assertions hold:*

(i) D_k^ν *meets* $I_{k'} \times \{\infty\}$ *with respect to the set* $R \times S_{k'}^{\nu'}$ *for some* $(k',\nu') \neq (k,\nu)$;

(ii) D_k^ν *meets* $R \times \{\tilde{0}\}$ *for some* $\lambda \in J_k \subset R$;

(iii) *the natural projection* $P_{R \times \{\tilde{0}\}}(D_k^\nu)$ *of* D_k^ν *onto* $R \times \{\tilde{0}\}$ *is unbounded.*

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