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SMOOTH REPRESENTABILITY OF A SMOOTH MULTIVARIATE FUNCTION BY SUMS OF GENERALIZED RIDGE FUNCTIONS

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Abstract

In this paper, we find a condition under which the smooth multivariate function can be represented by sums of smooth generalized ridge functions.

Keywords: ridge function, smooth multivariate function, representation, increment, generalized ridge function.
Mathematics Subject Classification *(2020):*

1. Introduction

A multivariate function $F: R^n \to R$ of the form

$$F(\mathbf{x}) = f\left(\mathbf{a}^1 \cdot \mathbf{x}, \ldots, \mathbf{a}^d \cdot \mathbf{x}\right)$$

is called a generalized ridge function, where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$, $f: R^d \to R$ is a real-valued function defined on R^d, $1 \le d < n$, $\mathbf{a}^j = \left(a_1^j, \ldots, a_n^j\right) \in R^n \setminus \{0\}$,

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$j=\overline{1,d}$ are fixed linearly independent vectors (directions).For $d=1$, generalized ridge function called a ridge function. Ridge functions and generalized ridge functions arise naturally in various fields. We refer the reader to the monographs of A.Pinkus [1] and V.Ismailov [2] for a detailed and systematic study of ridge functions.

Definition 1. Let $\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}$ and $\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\}$, $1\le d<n$, be linear independent vector systems in R^n. If

$$\mathbf{span}\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}=\mathbf{span}\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\}$$

then the systems $\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}$ and $\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\}$ are called eqiavalent, otherwise, that is, if

$$\mathbf{span}\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}\neq\mathbf{span}\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\},$$

then the systems $\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}$ and $\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\}$ are called non-eqiavalent.

Remark 1. Obviously, if the systems $\left\{\mathbf{a}^1,...,\mathbf{a}^d\right\}$ and $\left\{\mathbf{b}^1,...,\mathbf{b}^d\right\}$ are eqiavalent, then any generalized ridge function of the form

$$F(\mathbf{x})=f\left(\mathbf{a}^1\cdot\mathbf{x},\ldots,\mathbf{a}^d\cdot\mathbf{x}\right)$$

also has the form

$$F(\mathbf{x})=g\left(\mathbf{b}^1\cdot\mathbf{x},\ldots,\mathbf{b}^d\cdot\mathbf{x}\right).$$

Therefore, when defining a generalized ridge function, without loss of generality, we can assume that the vectors $\mathbf{a}^1,...,\mathbf{a}^d$ are unit and mutually perpendicular.

Assume we are given a function $F\in C^{(s)}\left(R^n\right)$ of the form

$$F(\mathbf{x})=\sum_{k=1}^{m}f_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,d}\cdot\mathbf{x}\right)$$

where $\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,d}\right\},...,\left\{\mathbf{a}^{m,1},...,\mathbf{a}^{m,d}\right\}$ are fixed pairwise non-eqiavalent vector systems in R^n and $f_k:R^d\to R$, $k=\overline{1,m}$ are the real-valued functions.

Is it true that there will always exist $g_k\in C^{(s)}\left(R^d\right)$, $k=\overline{1,m}$ such that

$$F(\mathbf{x})=\sum_{k=1}^{m}g_k\left(\mathbf{a}^{(k),1}\cdot\mathbf{x},\ldots,\mathbf{a}^{(k),d}\cdot\mathbf{x}\right)?$$

This question was posed in M.Buhmann and A.Pinkus [3] for ridge function representation and Pinkus [4] for generalized ridge function representation. In [7, 8, 9], the authors gave a partial solution to the above representation problem for ridge function representation. In [9], this problem for ridge function representation is solved up to a multivariate polynomial:

For the representation of generalized ridge function proved in [10] following theorem.

Theorem 1. Assume a function $F \in C^{(m)}\left(R^n\right)$ is of the form

$$F(\mathbf{x}) = \sum_{k=1}^{m} f_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right), \quad \mathbf{x}\in R^n,$$

where $\left\{\mathbf{a}^{1,1},\ldots,\mathbf{a}^{1,n-1}\right\},\ldots,\left\{\mathbf{a}^{m,1},\ldots,\mathbf{a}^{m,n-1}\right\}$ are fixed pairwise non-eqiavalent vector systems in R^n, $f_1,\ldots,f_m$ are arbitrarily behaved real-valued functions of $n-1$ variables. Then there exist functions $g_k \in C^{(1)}\left(R^{n-1}\right)$, $k=\overline{1,m}$, such that

$$F(\mathbf{x}) = \sum_{k=1}^{m} g_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right), \quad \mathbf{x}\in R^n. \tag{1}$$

İn this paper we show that if $m\geq 2$, then the representation of generalized ridge function (1) satisfied for $g_k \in C^{(2)}\left(R^{n-1}\right)$.

2.The smoothness problem in generalized ridge function representation

Lemma 1. Let $p\geq 0$ and

$$F(x_1,\ldots,x_n) = f_1(x_1,\ldots,x_{n-2},x_{n-1}) + f_2(x_1,\ldots,x_{n-2},x_n) \in C^{(p)}\left(R^n\right).$$

Then

$$f_1(x_1,\ldots,x_{n-2},x_{n-1}) - f_1(x_1,\ldots,x_{n-2},0) \in C^{(p)}\left(R^{n-1}\right).$$

Proof of Lemma 1. İndeed, it follows from equations

$$F(x_1,\ldots,x_n) = f_1(x_1,\ldots,x_{n-2},x_{n-1}) + f_2(x_1,\ldots,x_{n-2},x_n),$$

$$F(x_1,...,x_{n-2},0,x_n)=f_1(x_1,...,x_{n-2},0)+f_2(x_1,...,x_{n-2},x_n)$$

that

$$f_1(x_1,...,x_{n-2},x_{n-1})-f_1(x_1,...,x_{n-2},0)=$$

$$=F(x_1,...,x_n)-F(x_1,...,x_{n-2},0,x_d)\in C^{(p)}\left(R^{n-1}\right).$$

This comletes the proof of the Lemma.

Lemma 2. Let $p\geq 0$ and

$$F(x_1,...,x_n)=f_1(x_1,...,x_{n-2},x_{n-1})+f_2(x_1,...,x_{n-2},x_n)\in C^{(p)}\left(R^n\right).$$

Then there exist functions

$$g_1(x_1,...,x_{n-2},x_{n-1}),\ g_2(x_1,...,x_{n-2},x_n)\in C^{(p)}\left(R^{n-1}\right)$$

such that

$$F(x_1,...,x_n)=g_1(x_1,...,x_{n-2},x_{n-1})+g_2(x_1,...,x_{n-2},x_n).$$

Proof of Lemma 2. Let us denote

$$g_1(x_1,...,x_{n-2},x_{n-1})=f_1(x_1,...,x_{n-2},x_{n-1})-f_1(x_1,...,x_{n-2},0),$$

$$g_2(x_1,...,x_{n-2},x_n)=f_2(x_1,...,x_{n-2},x_n)+f_1(x_1,...,x_{n-2},0).$$

It follows from the Lemma 1 that $g_1\in C^{(p)}\left(R^{n-1}\right)$ and the following equation is satisfied

$$F(x_1,...,x_n)=f_1(x_1,...,x_{n-2},x_{n-1})+f_2(x_1,...,x_{n-2},x_n)=$$

$$=\left[f_1(x_1,...,x_{n-2},x_{n-1})-f_1(x_1,...,x_{n-2},0)\right]+$$

$$+\left[f_2(x_1,...,x_{n-2},x_n)+f_1(x_1,...,x_{n-2},0)\right]=$$

$$=g_1(x_1,...,x_{n-2},x_{n-1})+g_2(x_1,...,x_{n-2},x_n).$$

It follows that

$$g_2(x_1,...,x_{n-2},x_n)=F(x_1,...,x_n)-g_1(x_1,...,x_{n-2},x_{n-1})\in C^{(p)}\left(R^{n-1}\right).$$

This comletes the proof of the Lemma.

Lemma 3. Let $p\geq 0$ and $\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}$, $\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}$ is non-eqiavalent vector system in R^n. If

$$F(\mathbf{x})=f_1\left(\mathbf{a}^{1,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{1,n-1}\cdot\mathbf{x}\right)+f_2\left(\mathbf{a}^{2,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{2,n-1}\cdot\mathbf{x}\right), \qquad (2)$$

then there exist functions $g_1,g_2\in C^{(p)}\left(R^{n-1}\right)$, such that

$$F(\mathbf{x})=g_1\left(\mathbf{a}^{1,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{1,n-1}\cdot\mathbf{x}\right)+g_2\left(\mathbf{a}^{2,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{2,n-1}\cdot\mathbf{x}\right). \qquad (3)$$

Proof of Lemma 3. We assume that the system $\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}$, $\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}$ is non-equivalent vector system. Then there exist the unit vectors $\mathbf{c}^1,...,\mathbf{c}^{n-2}$ such that

$$\mathbf{span}\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}\cap\mathbf{span}\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}=\mathbf{span}\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2}\right\}.$$

Therefore, there exist the vectors $\mathbf{c}^{n-1}$ and $\mathbf{c}^n$ such that the following equations hold:

$$\mathbf{span}\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}=\mathbf{span}\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^{n-1}\right\},$$

$$\mathbf{span}\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}=\mathbf{span}\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^{n}\right\}.$$

İn other words, the systems $\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}$ and $\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^{n-1}\right\}$, $\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}$ and $\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^{n}\right\}$ are equivalent. Then we can write equation (2) in the form

$$F(\mathbf{x})=f_1\left(\mathbf{c}^1\cdot\mathbf{x},\ldots,\mathbf{c}^{n-2}\cdot\mathbf{x},\mathbf{c}^{n-1}\cdot\mathbf{x}\right)+f_2\left(\mathbf{c}^1\cdot\mathbf{x},\ldots,\mathbf{c}^{n-2}\cdot\mathbf{x},\mathbf{c}^{n}\cdot\mathbf{x}\right) \qquad (4)$$

İf we adopt the notation

$$\begin{cases} y_1 = \mathbf{c}^1 \cdot \mathbf{x} \\ \quad \cdots\cdots \\ y_n = \mathbf{c}^n \cdot \mathbf{x} \end{cases}$$

then the equation (4) can be rewritten as follows

$$\widetilde{F}(y_1,...,y_n) = F(\mathbf{x}) =$$

$$= \widetilde{f}_1(y_1,...,y_{n-2},y_{n-1}) + \widetilde{f}_2(y_1,...,y_{n-2},y_n) \in C^{(p)}\left(R^n\right).$$

According to Lemma 2, there exist functions $\widetilde{g}_1, \widetilde{g}_2 \in C^{(p)}\left(R^{n-1}\right)$ such that

$$\widetilde{F}(y_1,...,y_n) = \widetilde{g}_1(y_1,...,y_{n-2},y_{n-1}) + \widetilde{g}_2(y_1,...,y_{n-2},y_n).$$

Therefore, we have

$$F(\mathbf{x}) = \widetilde{F}(y_1,...,y_n) = \widetilde{g}_1(y_1,...,y_{n-2},y_{n-1}) + \widetilde{g}_2(y_1,...,y_{n-2},y_n) =$$

$$= \widetilde{g}_1\left(\mathbf{c}^1 \cdot \mathbf{x},...,\mathbf{c}^{n-2} \cdot \mathbf{x}, \mathbf{c}^{n-1} - 1 \cdot \mathbf{x}\right) + \widetilde{g}_2\left(\mathbf{c}^1 \cdot \mathbf{x},...,\mathbf{c}^{n-2} \cdot \mathbf{x}, \mathbf{c}^n \cdot \mathbf{x}\right). \qquad (5)$$

Since the vector systems $\left\{\mathbf{a}^{1,1},...,\mathbf{a}^{1,n-1}\right\}$ and $\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^{n-1}\right\}$, $\left\{\mathbf{a}^{2,1},...,\mathbf{a}^{2,n-1}\right\}$ and $\left\{\mathbf{c}^1,...,\mathbf{c}^{n-2},\mathbf{c}^n\right\}$ are equivalent, we can express the equation (5) of the form

$$F(\mathbf{x}) = g_1\left(\mathbf{a}^{1,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{1,n-1}\cdot\mathbf{x}\right) + g_2\left(\mathbf{a}^{2,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{2,n-1}\cdot\mathbf{x}\right).$$

This completes the proof of the Lemma 3.

Lemma 4. Let $\mathbf{a}^1,...,\mathbf{a}^{n-1}$ are any linearly independent vectors in R^n and the vector $l \in R^n$ is not perpendicular to the vector space $\mathbf{span}\left\{\mathbf{a}^1,...,\mathbf{a}^{n-1}\right\}$. Then for any function $\varphi(\mathbf{u}) = \varphi(u_1,...,u_{n-1}) \in C^{(2)}\left(R^{n-1}\right)$ there exist a

continuously differentiable generalized ridge function of the form $\Phi(\mathbf{x}) = f\left(\mathbf{a}^1 \cdot \mathbf{x},\ldots,\mathbf{a}^{n-1} \cdot \mathbf{x}\right)$ such that

$$\frac{\partial \Phi}{\partial l}(\mathbf{x}) = \varphi\left(\mathbf{a}^1 \cdot \mathbf{x},\ldots,\mathbf{a}^{n-1} \cdot \mathbf{x}\right)$$

for any $\mathbf{x} \in R^n$.

Proof of Lemma 4. It follows from Remark 1 that without loss of generality, we can assume that the vectors $\mathbf{a}^1,\ldots,\mathbf{a}^{n-1}$ are unit and mutually perpendicular. Denote by $\mathbf{a}^0$ the unit vector, perpendicular to the vectors $\mathbf{a}^1,\ldots,\mathbf{a}^{n-1}$. Let

$$l = \sum_{p=0}^{n-1} \alpha_p \mathbf{a}^p .$$

As the vector $l \in R^{\mathrm{n}}$ is not perpendicular to the vector space $\mathbf{span}\left\{\mathbf{a}^1,\ldots,\mathbf{a}^{n-1}\right\}$, then

$$\eta_0 = \sum_{p=1}^{n-1} \alpha_p^2 > 0 .$$

Denote

$$\Phi(\mathbf{x}) = \frac{1}{\eta_0} \int_0^{\sum_{p=1}^{n-1} \alpha_p \mathbf{a}^p \cdot \mathbf{x}} \varphi\left(s_1(t,\mathbf{x}),\ldots,s_{n-1}(t,\mathbf{x})\right) dt ,$$

where

$$s_k(t,\mathbf{x}) = \frac{1}{\eta_0}\left(\alpha_k t + \sum_{p=1,\, p\neq k}^{n-1} \left(\alpha_p^2 \cdot \mathbf{a}^k - \alpha_p \alpha_k \cdot \mathbf{a}^p\right) \cdot \mathbf{x}\right), \quad k = 1,\ldots,n-1 .$$

It follows from equations

$$\frac{\partial \Phi}{\partial \mathbf{a}^k}(\mathbf{x}) = \frac{\alpha_k}{\eta_0} \cdot \varphi\left(\mathbf{a}^1 \cdot \mathbf{x},\ldots,\mathbf{a}^{n-1} \cdot \mathbf{x}\right) +$$

$$+\frac{1}{\eta_0^2}\int_0^{\sum_{p=1}^{n-1}\alpha_p \mathbf{a}^p\cdot\mathbf{x}}\left(\sum_{j=1,\, j\neq k}^{n-1}\frac{\partial\varphi}{\partial u_j}\left(s_1(t,\mathbf{x}),\ldots,s_{n-1}(t,\mathbf{x})\right)\cdot\left(-\alpha_k\alpha_j\right)\right)dt,$$

$$+\frac{1}{\eta_0^2}\int_0^{\sum_{p=1}^{n-1}\alpha_p \mathbf{a}^p\cdot\mathbf{x}}\left(\frac{\partial\varphi}{\partial u_k}\left(s_1(t,\mathbf{x}),\ldots,s_{n-1}(t,\mathbf{x})\right)\cdot\sum_{p=1,\, p\neq k}^{n-1}\alpha_p^2\right)dt,\quad k=1,\ldots,n-1,$$

$$\frac{\partial\Phi}{\partial\mathbf{a}^0}(\mathbf{x})=0$$

that

$$\frac{\partial\Phi}{\partial l}(\mathbf{x})=\sum_{k=0}^{n-1}\alpha_k\frac{\partial\Phi}{\partial\mathbf{a}^k}(\mathbf{x})=\varphi\left(\mathbf{a}^1\cdot\mathbf{x},\ldots,\mathbf{a}^{n-1}\cdot\mathbf{x}\right).$$

On the other side, it follows from $\frac{\partial\Phi}{\partial\mathbf{a}^0}(\mathbf{x})=0$ that the function Φ is of the form $\Phi(\mathbf{x})=f\left(\mathbf{a}^1\cdot\mathbf{x},\ldots,\mathbf{a}^{n-1}\cdot\mathbf{x}\right)$. This completes the proof of the Lemma 4.

Theorem 2. Let $m\geq 2$ and the function $F\in C^{(m)}\left(R^n\right)$ is of the form

$$F(\mathbf{x})=\sum_{k=1}^{m}f_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right),\quad \mathbf{x}\in R^n. \tag{6}$$

where $\left\{\mathbf{a}^{1,1},\ldots,\mathbf{a}^{1,n-1}\right\},\ldots,\left\{\mathbf{a}^{m,1},\ldots,\mathbf{a}^{m,n-1}\right\}$ are fixed pairwise non-eqiavalent vector systems in R^n, $f_1,\ldots,f_m$ are arbitrarily behaved real-valued functions of $n-1$ variables. Then there exist functions $g_k\in C^{(2)}\left(R^{n-1}\right)$, $k=\overline{1,m}$, such that

$$F(\mathbf{x})=\sum_{k=1}^{m}g_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right),\quad \mathbf{x}\in R^n. \tag{7}$$

Proof of Theorem 2. We will prove the theorem using the method of mathematical induction. In the case $m=2$ theorem follows from Lemma 3. Assume that the theorem is true in the case $m-1$. We will show that it also holds for m. Since the function $F\in C^{(m)}\left(R^n\right)$ can be represented by (6), it follows

from Theorem 1 that there exist the functions $h_k \in C^{(1)}\left(R^{n-1}\right),\ k=1...m$, such that

$$F(\mathbf{x})=\sum_{k=1}^{m} h_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right). \tag{8}$$

Thus, we obtain that

$$\frac{\partial F}{\partial l_m}(\mathbf{x})=\sum_{k=1}^{m-1}\sum_{i=1}^{n-1}\mathbf{a}^{k,i}\cdot l_m\frac{\partial h_k}{\partial u_i}\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right), \tag{9}$$

where l_m is a unit vector, perpendicular to the vector systems $\left\{\mathbf{a}^{m,1},\ldots,\mathbf{a}^{m,n-1}\right\}$. It follows from (9) that

$$\frac{\partial F}{\partial l_m}(\mathbf{x})=\sum_{k=1}^{m-1}\tilde{h}_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right)\in C^{(m-1)}\left(R^{n}\right). \tag{10}$$

Then, by the principle of mathematical induction, we obtain that there exist functions $\tilde{g}_k \in C^{(2)}\left(R^{n-1}\right), k=1,\ldots,m-1$, such that

$$\frac{\partial F}{\partial l_m}(\mathbf{x})=\sum_{k=1}^{m}\tilde{g}_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right).$$

According to Lemma 4, there exist functions

$$\phi_k(\mathbf{x})=g_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right)\in C^{(2)}\left(R^{n-1}\right),\ k=\overline{1,m-1}$$

such that

$$\frac{\partial\phi_k}{\partial l_m}(\mathbf{x})=\tilde{g}_k\left(\mathbf{a}^{k,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{k,n-1}\cdot\mathbf{x}\right),\ \ k=\overline{1,m-1}.$$

Then it follows from (10) that we can write it in the form of

$$\frac{\partial}{\partial l_m}\left[F(\mathbf{x})-\sum_{k=1}^{m-1}\phi_k(\mathbf{x})\right]=0. \tag{11}$$

İt follows from (11) that the difference $F(\mathbf{x})-\sum_{k=1}^{m-1}\phi_k(\mathbf{x})$ does not depend on the direction l_m. Therefore there exist the function $g_m : R^{n-1} \to R$ such that

$$F(\mathbf{x})-\sum_{k=1}^{m-1}\phi_k(\mathbf{x}) = g_m\left(\mathbf{a}^{m,1}\cdot\mathbf{x},\ldots,\mathbf{a}^{m,n-1}\cdot\mathbf{x}\right). \tag{12}$$

Taking into account the expression of the functions, it follows that equation (7) is satisfied. Given the conditions $g_k \in C^{(2)}\left(R^{n-1}\right)$, $k=\overline{1,m-1}$ and $F \in C^{(m)}\left(R^n\right)$ it follows from (12) that $g_m \in C^{(2)}\left(R^{n-1}\right)$. This completes the proof of the Theorem 2.

References

[1] Pinkus A. *Ridge functions*. Cambridge Tracts in Mathematics, 205. Cambridge University Press; 2015.

[2] Ismailov VE. *Ridge Functions and Applications in Neural Networks*. AMS book series: Mathematical surveys and monographs, vol. 263; 2021.

[3] Buhmann MD, Pinkus A. Identifying linear combinations of ridge functions. *Adv Appl Math* **1999**, v. 22, p. 103-118.

[4] Pinkus A. Smoothness and uniqueness in ridge function representation. *Indag Math* **2013**, v. 24(4), p. 725-738.

[5] Aliev RA, Ismailov VE. On a smoothness problem in ridge function representation. *Adv Appl Math* **2016**, v. 73, p. 154-169.

[6] Kuleshov AA. On some properties of smooth sums of ridge functions (in Russian). *Tr Mat Inst Steklova* **2016**, v. 294, p. 99-104.

[7] Aliev RA, Asgarova AA, Ismailov VE. A note on continuous sums of ridge functions. *J Approx Theory* **2019**, v. 237, p. 210-221.

[8] Aliev RA, Ismailov VE. A representation problem for smooth sums of ridge functions. *J Approx Theory* **2020**, v. 257, no. 105448, 13 pp.

[**9]** Isgandarli FM**.** On the representability of a smooth trivariate function by sums of generalized ridge functions. *Baku State University Journal of Physics &*

*Space Sciences **2024**, v 1 (1), p. 16-26.*

[10] Akbarov AA**,** Isgandarli FM. On the representability of a smooth function by sums of generalized ridge functions.Proceeding of Institute of Mathematics and mechanics, National Academy of Sciences of Azerba