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AN OPTIMAL CONTROL PROBLEM FOR AN ELLIPTIC EQUATION WITH PERIODICITY CONDITIONS

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Abstract

An optimal control problem for an elliptic equation with periodic boundary condition is considered. The correctness of the problem statement is investigated. The theorem on the existence and uniqueness of the solution to the considered optimal control problem is proved. It is proved that the objective functional is differentiable and an explicit expression for its gradient is derived. Necessary and sufficient conditions for the optimality of the controls are obtained.

Keywords: elliptic equation, optimal control, periodicity conditions, correctness of problem statement, optimality criterion
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1. Introduction

Optimal control problems for elliptic equations arise in the fields of elasticity theory, heat transfer, convection-diffusion-reaction, and environmental forecasting [1-3]. Such problems have been studied quite thoroughly for elliptic

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equations with classical boundary conditions [4-11]. However, these problems have been less studied in the case of periodic boundary conditions.

2. Statement of the problem and its correctness

Let $\Omega = \{x = (x_1,..., x_n) \in R^n : 0 < x_1 < l_i,\ i = \overline{1,n}\}$ be a parallelepiped in R^n. Let us denote by $\hat{W}_2^1(\Omega)$ the subspace formed by the l - periodic elements of $W_2^1(\Omega)$, i.e. the elements $u = u(x) \in W_2^1(\Omega)$ satisfying the condition $u|_{x_i=0} = u|_{x_i=l_i},\ i = \overline{1,n}$. In this work, the scalar product in the space $L_2(\Omega)$ is denoted by (u,v), and the norm is denoted by the symbol $\|u\|$. Let us consider the following optimal control problem: İt is required to minimize the functional

$$J(v) = \int_\Omega |u(x,v) - z(x)|^2 dx + \alpha \int_\Omega v^2(x) dx \tag{1}$$

under the following conditions

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial u}{\partial x_i}\right) + \alpha(x)u = v(x), \quad x \in \Omega, \tag{2}$$

$$u|_{x_i=0} = u|_{x_i=l_i}, \qquad i = \overline{1,n}, \tag{3}$$

$$k(x)\frac{\partial u}{\partial x_i}\bigg|_{x_i=0} = k(x)\frac{\partial u}{\partial x_i}\bigg|_{x_i=l_i}, \qquad i = \overline{1,n}, \tag{4}$$

$$v = v(x) \in V \subseteq L_2(\Omega). \tag{5}$$

Here $\alpha \ge 0$ is a given number, $V \subseteq L_2(\Omega)$ is a given set, $v = v(x) \in V \subset L_2(\Omega)$ is a control, $k(x)$, $a(x)$, $z(x)$ are given measurable functions satisfying the following conditions

$$0 < \nu \le k(x) \le \mu, \quad 0 < \mu_1 \le a(x) \le \mu_2, \quad x \in \Omega,$$

$$\nu, \mu, \mu_1, \mu_2 = const, \quad z(x) \in L_2(\Omega). \tag{6}$$

Definition 1. The generalized solution to the boundary value problem (2)-(4) in the space $W_2^1(\Omega)$ for each fixed $v = v(x) \in L_2(\Omega)$ is called the function $u = u(x) = u(x,v) \in \hat{W}_2^1(\Omega)$ that it satisfies the following integral identity for $\forall \eta = \eta(x) \in \hat{W}_2^1(\Omega)$:

$$L(u,\eta) = \int_\Omega \left(\sum_{i=1}^n k(x) \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} + \alpha(x) u \eta \right) dx = \int_\Omega v(x) \eta dx. \tag{7}$$

Theorem 1. Assume that conditions (6) are satisfied. Then, for each fixed $v \in L_2(\Omega)$, there can be at most one generalized solution to the boundary value problem (2)-(4) in the space $W_2^1(\Omega)$ and the estimate

$$\|u\|_{W_2^1(\Omega)} \le M_1 \|v\| \tag{8}$$

is hold. Here $M_1 = (\mu_1 \mu_3)^{-\frac{1}{2}}$.

Proof. Consider the quadratic form $L(u,u)$. If we choose $\eta = u$ in (7) and taking into account the inequalities (6), we obtain

$$L(u,u) = \int_\Omega \sum_{i=1}^n \left[k(x) \left(\frac{\partial u}{\partial x_i} \right)^2 + \alpha(x) u^2 \right] dx \ge$$

$$\ge \int_\Omega \left(\nu u_x^2 + \mu_1 u^2 \right) dx = \nu \|u_x\|^2 + \mu_1 \|u\|^2 \tag{9}$$

here $u_x^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2$.

According to "Cauchy inequality with ε" [11, p.33] when $\varepsilon = \frac{\mu_1}{2}$, we obtain the following inequality:

$$L(u,u) = (v,u) \le \|v\| \|u\| \le \frac{\mu_1}{2} \|u\|^2 + \frac{1}{2\mu_1} \|v\|^2. \tag{10}$$

It follows from (9) and (10) that:

$$\nu\|u_x\|^2 + \frac{\mu_1}{2}\|u\|^2 \le \frac{1}{2\mu_1}\|v\|^2 .$$

Hence we obtain the following inequality:

$$\|u_x\|^2 + \|u\|^2 \le \frac{1}{\mu_1\mu_3}\|v\|^2.$$

Here $\mu_3 = \min(2\nu, \mu_1)$.

From here the estimation (8) is obtained. From estimate (8) it is clear that for $v = 0$ the solution $u(x)$ must be equal to zero, and, therefore, problem (2)-(4) can have no more than one generalized solution from $W_2^1(\Omega)$. The theorem is proved.

Theorem 2. If conditions (6) are satisfied, then for every $v \in L_2(\Omega)$ there exists a generalized solution to the problem (2)-(4) from the space $W_2^1(\Omega)$.

Proof. Let us define a new scalar product and norm in the space $\hat{W}_2^1(\Omega)$:

$$[u,v] = \int_\Omega \sum_{i=1}^n \left(k(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \alpha(x)uv \right)dx, \quad \|u\|_1 = \sqrt{(u,u)}.$$

This scalar product is identical to the following scalar product of the space $\hat{W}_2^1(\Omega)$:

$$(u,v)_{\hat{W}_2^1(\Omega)} = \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + uv \right)dx.$$

Therefore, the identity (7) can be represented as follows:

$$[u,\eta] = (v,\eta). \tag{11}$$

When $v \in L_2(\Omega)$ is specified, the expression (v,η) defines a linear functional depending on η in the space $\hat{W}_2^1(\Omega)$. Besides that, since

$$|(v,\eta)| \le \|v\|\|\eta\| \le M_2\|v\|\|\eta\|_1,$$

the functional is bounded. Here the constant $M_2 > 0$ does not depend on v and η. Then according to Ritz theorem[12, p.75], there is a unique function $F \in \hat{W}_2^1(\Omega)$ such that

$$[v,\eta] = [F,\eta], \quad \forall \eta \in \hat{W}_2^1(\Omega).$$

From this and from (11) it follows that, there is only one function $u = F$ in space $\hat{W}_2^1(\Omega)$ which satisfies the identity (7). The theorem is proved.

Theorem 3. Suppose that conditions (6) are satisfied and the set V is a closed, convex and bounded set in the space $L_2(\Omega)$. Then the optimal controls set V_* of problem (1)-(5) is not empty, but is closed, convex and bounded, and any minimizing sequence in the space $L_2(\Omega)$ converges weakly to the set V_*. Moreover, if $\alpha > 0$, then the set V_* consists in the only one point $v_* = v_*(x) \in V$, and the arbitrary minimizing sequence converges to the element v_* according to the norm of the space $L_2(\Omega)$.

Proof. Note that the functional (1) under conditions (2)-(4) is convex on $L_2(\Omega)$. This follows from the linearity of the solution $u = u(x,v)$ of the boundary value problem (2)-(4) for the control $v \in L_2(\Omega)$, and from the convexity of the function $|u - z|^2$ for $u \in R$. In addition, if $\alpha > 0$, then the functional (1) as the sum of a convex and a strongly convex functional is strongly convex on $L_2(\Omega)$.

Let us show that the functional (1) under conditions (2)-(4) is continuous on $L_2(\Omega)$. Let $v = v(x) \in L_2(\Omega)$ be some element, $\{v_k = v_k(x)\} \subset L_2(\Omega)$ be an arbitrary sequence, such that

$$v_k(x) \to v(x) \text{ strongly in } L_2(\Omega). \tag{12}$$

According to Theorems 1 and 2, each control $v_k = v_k(x) \in L_2(\Omega)$ corresponds to a unique generalized solution $u(x,v_k) \in \hat{W}_2^1(\Omega)$ of the boundary value problem (2)-(4). We denote $w_k(x) = u(x,v_k) - u(x,v)$, $x \in \Omega$. From (7) it follows that the identity is satisfied

$$\int_{\Omega}\left(\sum_{i=1}^{n} k(x)\frac{\partial w_k}{\partial x_i}\frac{\partial \eta}{\partial x_i}+\alpha(x)w_k\eta\right)dx=\int_{\Omega}\left(v_k(x)-v(x)\right)\eta dx,$$

$$\forall\,\eta=\eta(x)\in\hat{W}_2^1(\Omega).$$

From this it follows from Theorem 1 that the following estimate is valid:

$$\|z_k\|_{W_2^1(\Omega)}\le M_1\|v_k-v\|.$$

Then, by virtue of (12), we have

$$u(x,v_k)\to u(x,v)\ \text{strongly in}\ W_2^1(\Omega). \qquad (13)$$

From this and (12), (13) we obtain that $J(v_k)\to J(v)$ for $k\to\infty$, i.e. functional (1) is continuous on $L_2(\Omega)$. Then, by virtue of Theorem 5 from [13, p.52], functional (1) is weakly lower semicontinuous on $L_2(\Omega)$. In addition, the set V is convex, closed, and bounded in the space $L_2(\Omega)$. Therefore, the validity of Theorem 3 follows from [13, p.49, Theorem 2, p.51, Theorem 4; p.55, Theorem 8]. Theorem 3 is proved.

3. Differention of the objective functional and optimality criterion

Consider the adjoint problem corresponding to the problem (1)-(5):

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(k(x)\frac{\partial\psi}{\partial x_i}\right)+\alpha(x)\psi=2\left[u(x,v)-z(x)\right],\quad x\in\Omega, \qquad (14)$$

$$\psi\big|_{x_i=0}=\psi\big|_{x_i=l_i},\qquad i=\overline{1,n}, \qquad (15)$$

$$k(x)\frac{\partial\psi}{\partial x_i}\bigg|_{x_i=0}=k(x)\frac{\partial\psi}{\partial x_i}\bigg|_{x_i=l_i},\quad i=\overline{1,n}. \qquad (16)$$

Definition 2. The generalized solution to the boundary value problem (14)-(16) in the space $W_2^1(\Omega)$ for each fixed $v=v(x)\in L_2(\Omega)$ is called the function $\psi=\psi(x)=\psi(x,v)\in\hat{W}_2^1(\Omega)$ such that it satisfies the following integral identity for $\forall\,\eta=\eta(x)\in\hat{W}_2^1(\Omega)$:

$$\int_\Omega \left(\sum_{i=1}^n k(x) \frac{\partial \psi}{\partial x_i} \frac{\partial \eta}{\partial x_i} + \alpha \psi \eta \right) dx = 2 \int_\Omega [u(x,v) - z(x)] \eta dx. \quad (17)$$

From Theorems 1 and 2, it follows that for each $v \in L_2(\Omega)$ there exists a unique generalized solution to the problem (14)-(16) from the space $W_2^1(\Omega)$ and the following estimation is hold:

$$\|\psi\|_{W_2^1(\Omega)} \le 2M_1 [2M_1 \|v\| + \|z\|]. \quad (18)$$

Here $M_1 > 0$ is a constant.

Theorem 4. Let conditions (6) be satisfied. Then the functional (1) is Fréchet continuous differentiable in the space $L_2(\Omega)$ and its gradient at the point $v \in L_2(\Omega)$ is determined by the following equality

$$J'(v) = \psi(x,v) + 2\alpha v(x), \quad x \in \Omega. \quad (19)$$

Proof. Let us choose the controls $v, v + \Delta v \in L_2(\Omega)$ and assume that $\Delta u(x,v) = u(x, v + \Delta v) - u(x,v)$. Then from the conditions (2)-(4),it follows that Δu is the generalized solution to the following boundary problem:

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(k(x) \frac{\partial \Delta u}{\partial x_i} \right) + \alpha \Delta u = \Delta v, \quad x \in \Omega, \quad (20)$$

$$\Delta u_{x_i=0} = \Delta u_{x_i=l_i}, \quad i = \overline{1,n}, \quad (21)$$

$$k(x) \frac{\partial \Delta u}{\partial x_i} \bigg|_{x_i=0} = k(x) \frac{\partial \Delta u}{\partial x_i} \bigg|_{x_i=l_i}, \quad i = \overline{1,n}. \quad (22)$$

A generalized solution to this problem satisfies the following integral identity for arbitrary $\eta = \eta(x) \in \hat{W}_2^1(\Omega)$:

$$\int_\Omega \left(\sum_{i=1}^n k(x) \frac{\partial \Delta u}{\partial x_i} \frac{\partial \eta}{\partial x_i} + \alpha \Delta u \eta \right) dx = \int_\Omega \Delta v \eta dx. \quad (23)$$

The estimation (8) in Theorem 1 shows that

$$\|\Delta u\|_{W_2^1(\Omega)} \leq M_3 \|\Delta v\| \tag{24}$$

for a function Δu satisfying the identity (23). Here $M_3 > 0$ is a constant which does not depend on Δv.

Now let us consider the increment of the functional (1):

$$\Delta J(v) = J(v+\Delta v) - J(v) = \int_\Omega |u(x, v+\Delta v) - z(x)|^2 dx - \int_\Omega |u(x,v) - z(x)|^2 dx +$$

$$+\alpha \int_\Omega (v+\Delta v)^2 dx - \alpha \int_\Omega v^2 dx = 2\int_\Omega [u(x,v) - z(x)]\Delta u(x,v) dx + \int_\Omega |\Delta u(x,v)|^2 dx +$$

$$+\alpha \int_\Omega (2v\Delta v + \Delta v^2) dx. \tag{25}$$

If we choose $\eta = \Delta u$ in the identity (17) and $\eta = \psi$ in the identity (23) and subtract the resulting equations side by side, we obtain the following equation

$$2\int_\Omega [u(x,v) - z(x)]\Delta u dx = \int_\Omega \psi \Delta v dx.$$

Substituting this equation into (24), we obtain

$$\Delta J(v) = \int_\Omega (\psi + 2\alpha v)\Delta v dx + R, \tag{26}$$

where

$$R = \int_\Omega |\Delta u(x,v)|^2 dx + -\alpha \int_\Omega \Delta v^2 dx.$$

From the estimation (24) and the expression of R, we obtain that $R = o(\|\Delta v\|)$. It follows from (26) that the functional $J(v)$ is differentiable in the space $L_2(\Omega)$ and its gradient at the point $v \in L_2(\Omega)$ is determined by the equality (19). Using identity (19) and inequality (18), it can be show that the functional $J'(v)$ is continuous in $L_2(\Omega)$. The theorem is proved.

The following theorem expresses the optimality criterion for the problem (1)-(5) and its accuracy is derived from the sign of optimality for smooth convex functionals[13, p.28].

Theorem 5. The satisfaction of the following inequality is necessary and sufficient for the optimality of control $v_* = v_*(x) \in V$ in problem (1)-(5):

$$\int_\Omega (\psi_* + 2\alpha v_*)(v - v_*)dx \geq 0, \tag{27}$$

$$\forall v = v(x) \in V.$$

Here, the function $\psi_* = \psi(x, v_*)$ is the solution to the problem (14)-(16) corresponding to the control $v = v_*$. If $v_* \in \operatorname{int} V$, then the inequality (27) is equivalent to the equality

$$\psi_*(x) + 2\alpha v_*(x) = 0, \quad x \in \Omega.$$

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