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ON THE LAW OF LARGE NUMBERS FOR THE OF MARKOV RANDOM WALKS DESCRIBED BY THE AUTOREGRESSIVE PROCESS $AR(1)$

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Abstract

In this paper is proved the law of large numbers for the Markov random walks, discribed by the first-order autoregressive process ($AR(1)$).

Keywords: Markov random walk, first-order autoregressive process, the law of lage numbers.
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1. Introduction

It is known that the first-order autoregressive process ($AR(1)$) is determined by the solution of a recurrent equation of the form

$$X_n = \beta X_{n-1} + \xi_n \tag{1}$$

where $n \geq 1$, $\beta \in R = (-\infty, \infty)$ is some fixed number and the innovation $\{\xi_n\}$ is the sequence of independent identically distributed random variables with finite variance $\sigma^2 = D\xi_1 < \infty$ and with mean $a = E\xi_1$. It is assumed that the initial value of the process X_0 is independent on the innovation $\{\xi_n\}$.

The process $AR(1)$ plays a great role in theoretical and applied terms in the theory of Markov random walks ([1]- [10]).

The following Markov random walks are described by means of the process $AR(1)$

$$S_n = \sum_{k=0}^{n} X_k,$$

$$C_n = \sum_{k=1}^{n} X_k X_{k-1},$$

$$D_n = \sum_{k=1}^{n} X_{k-1}^2,$$

$$\theta_n = \frac{C_n}{D_n},$$

$$Z_n = \frac{C_n^2}{D_n},$$

$$H_n = \sum_{k=1}^{n} X_{k-1}\xi_x, \quad n \geq 1$$

These Markov random walks have been considered in the some problems of theory of nonlinear renewal theory and of sequential analysis ([1]- [10]) .

The limits theorems for the Markov random C_n, D_n, θ_n and Z_n are proved in the case $a = 0$ in works [1], [2], [4].

In the present paper, we prove the law of large numbers for the mentioned Markov random walks in general case when $a = E\xi_1 \in R = (-\infty, \infty)$.

Note that in many problems of theory of Markov random walks described by the process AR(1), the case $a \neq 0$ is more complicated compared in case $a = 0$. As noted in the works [8, 9] the case $a \neq 0$ has been studied much less. A number of statistical problems for the model (1.1), in the case $a \neq 0$ were studied in [6] and [7].

We have

Theorem. Let $EX_0^2 < \infty$, $|\beta| < 1$, and $\sigma^2 = D\xi_1 < \infty$. Then as $n \to \infty$ the following convergences in probability are satisfied:

1) $\frac{S_n}{n} \xrightarrow{P} \frac{a}{1-\beta}$;

2) $\frac{H_n}{n} \xrightarrow{P} \frac{a^2}{1-\beta}$;

3) $\frac{D_n}{n} \xrightarrow{P} \frac{\sigma^2}{1-\beta^2} + \left(\frac{a}{1-\beta}\right)^2$;

4) $\frac{C_n}{n} \xrightarrow{P} \frac{\beta\sigma^2}{1-\beta^2} + \left(\frac{a}{1-\beta}\right)^2$.

Proof. Let us prove statement 1). From (1) we find

$$\sum_{k=1}^{n} X_k = \beta \sum_{k=1}^{n} X_{k-1} + \sum_{k=1}^{n} \xi_k \tag{2}$$

Hence, taking into account

$$\sum_{k=1}^{n} X_{k-1} = \beta \sum_{k=1}^{n} X_k + (X_0 - X_n)$$

from (2) we have

$$(1-\beta)\sum_{k=1}^{n} X_k = \beta(X_0 - X_n) + \sum_{k=1}^{n} \xi_k$$

or

$$(1-\beta)\frac{S_n}{n} = \frac{\beta(X_0 - X_n)}{n} + \frac{1}{n}\sum_{k=1}^{n} \xi_k \tag{3}$$

By Markov inequality it follows from $E|X_0| < \infty$ that

$$\frac{X_0}{n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{4}$$

Prove that

$$\frac{X_n}{n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{5}$$

By sequential iterations it is easy to obtain from (1) the following representation for X_n

$$X_n = \beta^n X_0 + \sum_{k=0}^{n-1} \beta^k \xi_{n-k} \ . \tag{6}$$

From (6) by virtue of $b = E|\xi_1| < \infty$ we obtain

$$E|X_n| \le |\beta|^n E|X_0| + \sum_{k=0}^{n-1} |\beta|^k E|\xi_{n-k}| \le$$

$$\le E|X_0| + b\sum_{k=0}^{\infty} |\beta|^k = E|X_0| + \frac{b}{1-|\beta|} < \infty \ . \tag{7}$$

(5) follows from (7).

By the stroung low of large numbers, for random variables ξ_n we have

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \xrightarrow{a.s.} a, \text{ as } n \to \infty. \tag{8}$$

Thus, from (3), (4), (5) and (8) we have

$$\frac{1}{n} S_n \xrightarrow{P} \frac{a}{1-\beta}, \text{ as } n \to \infty.$$

To prove the statement 2), at first we prove that

$$\frac{\sum_{k=1}^{n} X_{k-1}(\xi_k - a)}{n} \xrightarrow{P} 0, \text{ as } n \to \infty. \tag{9}$$

To prove (9), it suffius to show

$$J = E\left(\frac{\sum_{k=1}^{n} X_{k-1}(\xi_k - a)}{n}\right)^2 \xrightarrow{P} 0, \text{ as } n \to \infty. \tag{10}$$

By virtue of independence of random variables ξ_k and X_{k-m}, $1 \le m \le k$, we have

$$J = \frac{1}{n^2} E\left(\sum_{k=1}^{n} X_{k-1}(\xi_k - a)\right)^2 = \frac{1}{n^2}\sum_{k=1}^{n} E\big(X_{k-1}(\xi_k - a)\big)^2 =$$

$$= \frac{1}{n^2}\sum_{k=1}^{n} EX_{k-1}^2 E(\xi_k - a)^2 = \frac{\sigma^2}{n^2}\sum_{k=1}^{n} EX_{k-1}^2. \tag{11}$$

We now prove that for ruther large n

$$\sum_{k=1}^{n} EX_{k-1}^2 = O(n). \tag{12}$$

From the representation (6) we can obtain

$$EX_n = \beta^n EX_0 + a\sum_{k=0}^{n-1}\beta^k \to \frac{a}{1-\beta} \tag{13}$$

as $n \to \infty$, since $|\beta| < 1$ and $E|X_0| < \infty$. .

Furthermore,

$$DX_n = E\left(\beta^n (X_0 - EX_0) + \sum_{k=0}^{n-1}\beta^k (\xi_{n-k} - a)\right)^2 =$$

$$= \beta^{2n} E|X_0 - EX_0|^2 + \sigma^2 \sum_{k=0}^{n-1}\beta^{2k} \to \frac{\sigma^2}{1-\beta^2} \tag{14}$$

as $n \to \infty$, since $E|X_0 - EX_0|^2 < \infty$.

From (13) and (14) it follows that

$$EX_n^2 \to \frac{\sigma^2}{1-\beta^2} + \left(\frac{a}{1-\beta}\right)^2 \text{ as } n \to \infty. \tag{15}$$

Consequently (12) follows from (15).
(10) follows from (11) and (12).
Thus, the convergence of (9) is proved.

Now, by virtue of the equality

$$\frac{H_n}{n} = \frac{\sum_{k=1}^{n} X_{k-1}(\xi_k - a)}{n} + \frac{a}{n}\sum_{k=1}^{n} X_{k-1}$$

and from (9) and statement 1) we obtain statement 2) of Theorem 1.

Let us prove statement 3). From (1) we have

$$\sum_{k=1}^{n} X_k^2 = \beta^2 \sum_{k=1}^{n} X_{k-1}^2 + 2\beta \sum_{k=1}^{n} X_{k-1}\xi_k + \sum_{k=1}^{n}\xi_k^2.$$

Hence we obtain

$$\left(1-\beta^2\right)\sum_{k=1}^{n} X_{k-1}^2 = X_0^2 - X_n^2 + 2\beta\sum_{k=1}^{n} X_{k-1}\xi_k + \sum_{k=1}^{n}\xi_k^2$$

or

$$\left(1-\beta^2\right)\frac{D_n}{n} = \frac{X_0^2 - X_n^2 + 2}{n} + 2\beta\frac{H_n}{n} + \frac{1}{n}\sum_{k=1}^{n}\xi_k^2. \qquad (16)$$

It is clear that that

$$\frac{X_0}{\sqrt{n}} \xrightarrow{P} 0, \text{ as } n \to \infty$$

and from estimate (7) we have

$$\frac{X_n}{\sqrt{n}} \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Then, by virtue of the statement 2).

By the strong law of large numbers for random variables ξ_n^2

$$\frac{1}{n}\sum \xi_k^2 \xrightarrow{a.s.} \sigma^2 + a^2$$

from (16) we obtain

$$\left(1-\beta^2\right)\frac{D_n}{n} \xrightarrow{P} \frac{2\beta a^2}{1-\beta} + \sigma^2 + a^2 = \sigma^2 + \frac{a^2(1+\beta)}{1-\beta}..$$

This implies statement 3) of the theorem.

To prove statement 4). We have

$$C_n = \sum_{k=1}^{n} X_{k-1}X_k = \sum_{k=1}^{n} X_{k-1}\left(\beta X_{k-1} + \xi_k\right) =$$

$$= \beta\sum_{k=1}^{n} X_{k-1}^2 + \sum_{k=1}^{n} X_{k-1}\xi_k$$

or

$$\frac{C_n}{n} = \frac{\beta D_n}{n} + \frac{H_n}{n}.$$

Hence, from statement 2) and 1) we obtain

$$\frac{C_n}{n} \xrightarrow{P} \beta\left(\frac{\sigma^2}{1-\beta^2} + \left(\frac{a}{1-\beta}\right)^2\right) + \frac{a^2}{1-\beta} = \frac{\beta\sigma^2}{1-\beta^2} + \left(\frac{a}{1-\beta}\right)^2.$$

Thus, the theorem is proved.

The following corollary follows from this theorem.

Corollary 2.1. Let the conditions of the theorem are satisfied, and $a = 0$, then

1) $\theta = \frac{C_n}{D_n} \xrightarrow{P} \beta$, as $n \to \infty$,

2) $\frac{X_n}{n} \xrightarrow{P} \frac{\sigma^2\beta^2}{1-\beta^2}$, as $n \to \infty$.

Corollary 2.2. Under the conditions of the theorem, we have

$$\beta_n = \frac{\sum_{k=1}^{n} (X_k - a) X_{k-1}}{D_n} \xrightarrow{P} \beta, \text{ as } n \to \infty,$$

The statement of corollary 2.2 follows directly from statements 3) and 4) of the theorem. The statements of corollary 2.3 follows by virtue of the equality

$$\beta_n = \frac{C_n}{n} - a\frac{S_{n-1}}{D_n}$$

from statements 1), 3) and 4) of the theorem.

In corollary 2.2 β_n is the least-squares estimator by the results of observations $X_0, X_1, X_2, \ldots, X_n$, and the case of $a = 0$ we have $\beta_n = \theta_n$.

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