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ON COMPLETENESS OF EIGENFUNCTIONS OF A DIFFERENTIAL OPERATOR WITH A CONJUGATION CONDITIONS AND A SUMMABLE POTENTIAL

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Abstract

In this paper is studied the spectral problem for a discontinuous second order differential operator with a summabl potential function and a spectral parameter in conjugation conditions, that arises by solving the problem on vibrations of a loaded string with free ends. In the case of a summable potential function, using abstract theorems on the stability of basis properties of multiple systems in a Banach space with respect to certain transformations, as well as theorems on the completeness and minimality of the system of eigenfunctions of the spectral problem in the spaces $L_p \oplus C$ and L_p.

Keywords: spectral problem; eigenfunctions; completeness, minimality

1. Introduction

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Consider the following spectral problem with a point of discontinuity:

$$l(y) = -y''(x) + q(x)y = \lambda y, \quad x \in \left(0; \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{aligned} & y'(0) = y'(1) = 0, \\ & y\left(\frac{1}{3} - 0\right) = y\left(\frac{1}{3} + 0\right), \\ & y'\left(\frac{1}{3} - 0\right) - y'\left(\frac{1}{3} + 0\right) = \lambda m y\left(\frac{1}{3}\right). \end{aligned}\right\} \tag{2}$$

here, λ is spectral parameter, $q(x)$is a complex-valued function summing over the interval (0,1), m is complex nuber, and $m \neq 0$. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method [1-3]. The case of boundary conditions corresponding to a string with fixed ends (i.e. when instead of the boundary conditions $y'(0) = y'(1) = 0$ in (2) $y(0) = y(1) = 0$ are taken), is investigated in [4-10]. In [11], the asymptotic expressions for the eigenvalues and eigenfunctions of problem (1)–(2) in the case $q(x)$ were obtained, a linearization operator was constructed, and theorems on completeness and minimality were rigorously established. Furthermore, [12,13] in the case $q(x)$ investigated the basis properties of the eigenfunctions of this problem in the spaces $L_p(0,1) \oplus C$ and Morrey spaces, respectively.

This work is a continuation of [14] and investigates the completeness and minimality of the system of eigenfunctions of the spectral problem (1)-(2) in the spaces $L_p(0,1)$ and $L_p(0,1) \oplus C$.

2. Necessary information and preliminary results

The spectral problem (1)-(2) has two series of eigenvalues, which are of the form $\lambda_{i,n} = \rho_{i,n}^2$, $i = 1,2;\ n \in Z^+$, $Z^+ = N \cup \{0\}$ where the numbers $\rho_{i,n}$ are the zeros of the characteristic determinant $\Delta(\rho)$ and $\Delta(\rho)$ is defined as follows:

$$\Delta(\rho) = \det\left\| U_{\nu,j}\left(y_{\nu,k}\right)\right\|, \; j,k = 1,2; \nu = \overline{1,4}. \tag{3}$$

Here, $y_{11}(x)$ and $y_{12}(x)$ constitute a fundamental system of solutions for equation (1) on the interval $\left[0, \frac{1}{3}\right]$, while $y_{21}(x)$ and $y_{22}(x)$ form the fundamental system of solutions for the same equation on the interval $\left[\frac{1}{3}, 1\right]$ and

According to [15], they are in the following asymptotic forms:

$$y_{11}(x) = e^{i\rho x}[1],\ y_{12}(x) = e^{-i\rho x}[1],\ x \in \left[0, \frac{1}{3}\right];$$

$$y_{21}(x) = e^{i\rho\left(x-\frac{1}{3}\right)}[1],\ y_{22}(x) = e^{-i\rho\left(x-\frac{1}{3}\right)}[1],\ x \in \left[\frac{1}{3}, 1\right].$$

Here and throughout the paper, the notation $[a] = a + O\left(\frac{1}{\rho}\right)$ is employed.

The following theorems have been proven in [14].

Theorem 1. *The eigenvalues of problem (1)-(2) are asymptotically simple and consist of two series:* $\lambda_{i,n} = \rho_{i,n}^2,\ i = 1,2;\ n \in Z^+,\ Z^+ = N \cup \{0\}$ and the following asymptotic expressions hold for $\rho_{i,n}$.

$$\begin{cases} \rho_{1,n} = 3\pi n + \dfrac{3\pi}{2} + O\left(\dfrac{1}{n}\right) \\ \rho_{2,n} = \dfrac{3\pi n}{2} + \dfrac{3\pi}{4} + O\left(\dfrac{1}{n}\right) \end{cases}.$$

Teorem 2. *Suppose that the function* $q(x)$ *satisfies the conditions of Theorem 1. Then, for the eigenvalues* $\lambda_{i,n} = \rho_{i,n}^2,\ i = 1,2;\ n \in Z^+,\ Z^+ = N \cup \{0\}$ *the corresponding eigenfunctions* $y_{i,n}(x)$ *satisfy the following asymptotic formulas:*

$$y_{1,n} = \begin{cases} \cos\left(3\pi n + \dfrac{3\pi}{2}\right)x + O\left(\dfrac{1}{n}\right), & x \in \left[0, \dfrac{1}{3}\right], \\ O\left(\dfrac{1}{n}\right), & x \in \left[\dfrac{1}{3}; 1\right], \end{cases}$$

$$y_{2,n} = \begin{cases} O\left(\dfrac{1}{n}\right), & x \in \left[0, \dfrac{1}{3}\right], \\ \cos\left(\dfrac{3\pi n}{2} + \dfrac{3\pi}{4}\right)(1-x) + O\left(\dfrac{1}{n}\right), & x \in \left[\dfrac{1}{3}; 1\right]. \end{cases}$$

Additionally, the following formula for $\Delta(\rho)$ is derived in [14]:

$$\Delta(\rho)=\rho\sin\frac{\rho}{3}\left(\beta_1+\beta_2\sin^2\frac{\rho}{3}+O\left(\frac{1}{\rho}\right)\right)+ \\ +\rho^2 m\cos\frac{\rho}{3}-2\rho^2 m\cos^3\frac{\rho}{3}+O\left(\frac{1}{\rho}\right). \tag{4}$$

Here, $\beta_1=-3+2mq_1-mq_2,\ \beta_2=4+2mq_1-2mq_2.$
We will use the following known inequalities

$$|\sin\rho|\le ce^{|\rho|\sin\varphi}, |\cos\rho|\le ce^{|\rho|\sin\varphi}, \tag{5}$$

where $\rho=re^{i\varphi},\ 0\le\varphi\le\pi.$ Besides, outside of circles of the same radius δ with centres in zero of $\sin\rho$ the following estimation is true

$$|\sin\rho|\ge m_\delta e^{r\sin\varphi}. \tag{6}$$

From estimations (5) , (6) and from the formula (4) it follows that at great values of $|\rho|$ outside circles $K_{j,n}(\delta)=\{\rho:|\rho-\rho_{j,n}|<\delta\}$ of radius δ with the centres in zero of $\Delta(\rho)$ the following estimation is true.

$$|\Delta(\rho)|\ge M_\delta re^{r\sin\varphi}. \tag{7}$$

3. Construction of the Green's function and the resolvents of the linearized operator

Now let's pass to construction of the Green's function of problem (1)- (2). İt is defined as a kernel of integral representation for solution of the corresponding non-homogeneous problem

$$-y''+q(x)y=\rho^2 y+f(x), \tag{8}$$

satisfying boundary conditions (2). The solution of problem (8),(2) will be sought in the form

$$y(x)=\begin{cases} y_1(x),\ for\ x\in\left[0,\frac{1}{3}\right], \\ y_2(x),\ for\ x\in\left[\frac{1}{3},1\right], \end{cases} \tag{9}$$

where

$$\begin{cases} y_1(x) = c_{11}y_{11}(x) + c_{12}y_{12}(x) + \int_0^{\frac{1}{3}} g(x,\xi,\rho) f(\xi) d\xi, \ \ x \in \left[0, \frac{1}{3}\right], \\ y_2(x) = c_{21}y_{21}(x) + c_{22}y_{22}(x) + \int_{\frac{1}{3}}^{1} g(x,\xi,\rho) f(\xi) d\xi, \ x \in \left[\frac{1}{3}, 1\right]. \end{cases} \tag{10}$$

$$g_1(x,\xi,\rho) = \begin{cases} -\frac{1}{2i\rho}\left(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}\right), & 0 \le x < \xi \le \frac{1}{3}, \\ \frac{1}{2i\rho}\left(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}\right), & 0 \le x < \xi \le \frac{1}{3}, \end{cases} \tag{11}$$

$$g_2(x,\xi,\rho) = \begin{cases} -\frac{1}{2i\rho}\left(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}\right), & \frac{1}{3} \le x < \xi \le 1, \\ \frac{1}{2i\rho}\left(e^{i\rho(x-\xi)} - e^{-i\rho(x-\xi)}\right), & \frac{1}{3} \le x < \xi \le 1. \end{cases} \tag{12}$$

Let us require that the function (9) satisfies the boundary conditions (2). Then for definition of numbers $C_{j,k}$ we obtain the system of algebraic equations

$$\begin{gathered} U_\nu(y) = \sum_{j,k=1}^{2} C_{j,k} U_{\nu,j}\left(y_{jk}\right) + \int_0^{\frac{1}{3}} U_{\nu 1}(g) f(\xi) d\xi + \\ + \int_{\frac{1}{3}}^{1} U_{\nu 2}(g) f(\xi) d\xi = 0, \nu = \overline{1,4} \end{gathered} \tag{13}$$

Let's define numbers $C_{j,k}$ from (13) and substituting their values in (10), for solving the problem (8), (10) we obtain the formula

$$y_1(x) = \int_0^{\frac{1}{3}} G_{11}(x,\xi,\rho) f(\xi) d\xi + \int_{\frac{1}{3}}^{1} G_{12}(x,\xi,\rho) f(\xi) d\xi, \ \ x \in \left[0, \frac{1}{3}\right],$$

$$y_1(x)=\int_0^{\frac{1}{3}} G_{11}(x,\xi,\rho)\, f(\xi)d\xi+\int_{\frac{1}{3}}^{1} G_{12}(x,\xi,\rho)\, f(\xi)d\xi,\ \ x\in\left[0,\frac{1}{3}\right],$$

where

$$G_{11}(x,\xi,\rho)\, f(\xi)d\xi=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & y_{11} & y_{12} & 0 & 0 \\ U_{\nu 1}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{12}(x,\xi,\rho)\, f(\xi)d\xi=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & y_{11} & y_{12} & 0 & 0 \\ U_{\nu 2}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{21}(x,\xi,\rho)\, f(\xi)d\xi=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & 0 & 0 & y_{21} & y_{22} \\ U_{\nu 1}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{22}(x,\xi,\rho)\, f(\xi)d\xi=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & 0 & 0 & y_{21} & y_{22} \\ U_{\nu 1}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$\begin{cases} U_{11}(g)=-\frac{1}{2}\left(e^{i\rho\xi}[1]+e^{-i\rho\xi}[1]\right), \quad U_{12}(g)=0, \quad U_{21}(g)=0, \\ U_{22}(g)=\frac{1}{2}\left(e^{i\rho(1-\xi)}[1]+e^{-i\rho(1-\xi)}[1]\right), U_{31}(g)=\frac{1}{2i\rho}\left(e^{i\rho\left(\frac{1}{3}-\xi\right)}[1]-e^{-i\rho\left(\frac{1}{3}-\xi\right)}[1]\right) \\ U_{32}(g)=\frac{1}{2i\rho}\left(e^{i\rho\left(\frac{1}{3}-\xi\right)}[1]-e^{-i\rho\left(\frac{1}{3}-\xi\right)}[1]\right), U_{41}(g)=\frac{1}{2i\rho}\left(e^{i\rho\left(\frac{1}{3}-\xi\right)}[1]+e^{-i\rho\left(\frac{1}{3}-\xi\right)}[1]\right) \\ U_{42}(g)=\frac{1}{2}\left(e^{i\rho\left(\frac{1}{3}-\xi\right)}[1]+e^{-i\rho\left(\frac{1}{3}-\xi\right)}[1]\right)+\frac{\rho^2 m}{2i\rho}\left(e^{i\rho\left(\frac{1}{3}-\xi\right)}[1]-e^{-i\rho\left(\frac{1}{3}-\xi\right)}[1]\right) \end{cases} \tag{14}$$

Let's substitute (13), (3) and (4) in determinants of formula for $G_{kj}(x,\xi,\rho)$. Transforming the received determinants similar to [15, p. 95], and then opening them, we obtain the formula for the Green's function components. We'll formulate it as a lemma.

Lemma 1. *For the Green's function components* $G_{kj}(x,\xi,\rho)$ *of the problem (1), (2) the following expressions are true:*

$$G_{11}(x,\xi,\rho)=\pm\rho^2 e^{i\rho(x-\xi)}[1]-\frac{\rho^2}{\Delta(\rho)}\left(2e^{-i\rho}[1]-i\rho m e^{-i\rho}[1]-i\rho m e^{-\frac{1}{3}i\rho}[1]\right)e^{i\rho x}e^{i\rho\xi}-$$

$$-\frac{\rho^2}{\Delta(\rho)}\left(2e^{i\rho}[1]+i\rho m e^{-\frac{i\rho}{3}}[1]+i\rho m e^{i\rho}[1]\right)e^{i\rho x}e^{-i\rho\xi}[1]-$$

$$-\frac{\rho^2}{\Delta(\rho)}\left(2e^{i\rho}[1]+2i\rho m e^{-\frac{i\rho}{3}}[1]+i\rho m e^{i\rho}[1]\right)e^{-i\rho x}e^{-i\rho\xi}[1]- \qquad (15)$$

$$-\frac{\rho^2}{\Delta(\rho)}\left(2e^{i\rho}[1]+i\rho m e^{i\rho}[1]\right)e^{-i\rho x}e^{-i\rho\xi}[1], \quad x\in\left[0,\frac{1}{3}\right], \xi\in\left[\frac{1}{3},1\right],$$

$$G_{12}(x,\xi,\rho)=2e^{-\frac{2}{3}i\rho}e^{-i\rho x}e^{-i\rho(\xi-1)}[1]\left(e^{\frac{2i\rho}{3}}[1]-e^{-i\rho\left(\xi-\frac{1}{3}\right)}e^{i\rho(\xi-1)}[1]\right)\left(e^{2i\rho x}[1]+1\right) \qquad (16)$$

$$G_{21}(x,\xi,\rho)=2e^{-i\rho}e^{-i\rho\left(\xi-\frac{1}{3}\right)}e^{-i\rho\left(x-\frac{1}{3}\right)}[1]\left(e^{\frac{i\rho}{3}}e^{i\rho\left(\xi-\frac{1}{3}\right)}e^{i\rho x}[1]+1\right)\left(e^{\frac{4i\rho}{3}}[1]+e^{2i\rho\left(x-\frac{1}{3}\right)}[1]\right) \qquad (17)$$

$$G_{22}(x,\xi,\rho)=\pm\rho^2 e^{i\rho(x-\xi)}[1]+\frac{\rho^2}{\Delta(\rho)}\left((1-i\rho m)e^{\frac{2i\rho}{3}}[1]-(1+i\rho m)e^{\frac{4}{3}i\rho}[1]+e^{\frac{-2i\rho}{3}}[1]\right)\times$$

$$\times e^{i\rho\left(x-\frac{1}{3}\right)}e^{-i\rho\xi}[1]+\frac{\rho^2}{\Delta(\rho)}\left(e^{-\frac{4i\rho}{3}}[1]-(1+i\rho m)e^{-\frac{2i\rho}{3}}[1]-e^{\frac{4i\rho}{3}}[1]\right)e^{i\rho\left(x-\frac{1}{3}\right)}e^{i\rho\xi}[1]-$$

$$-\frac{\rho^2}{\Delta(\rho)}\left(-(1+i\rho m)e^{\frac{2i\rho}{3}}[1]-(-3+i\rho m)e^{\frac{4i\rho}{3}}[1]\right)e^{-i\rho\left(x-\frac{1}{3}\right)}e^{-i\rho\xi}[1]- \qquad (18)$$

$$-\frac{\rho^2}{\Delta(\rho)}\left((1+i\rho m)-(1+i\rho m)e^{\frac{2i\rho}{3}}[1]\right)e^{-i\rho\left(x-\frac{1}{3}\right)}e^{i\rho\xi}[1], \quad x\in\left[0,\frac{1}{3}\right], \xi\in\left[\frac{1}{3},1\right],$$

Let us now proceed to the construction of linearizing operator. By

$W_p^k\left(0,\frac{1}{3}\right)\oplus W_p^k\left(\frac{1}{3},1\right)$ we denote a space functions whose contractions on segments $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},1\right]$ belong correspondingly to Sobolev spaces $W_p^k\left(0,\frac{1}{3}\right)$ and $W_p^k\left(\frac{1}{3},1\right)$. Let's define the operator L in $L_p(0,1)\oplus C$ as follows:

$$D(L)=\begin{cases}\hat{u}=L_p(0,1)\oplus C:\hat{u}=\left(u,mu\left(\frac{1}{3}\right)\right),u\in W_p^2\left(0,\frac{1}{3}\right)\oplus W_p^2\left(\frac{1}{3},1\right),\\ u'(0)=u'(1)=0,u\left(\frac{1}{3}-0\right)=u\left(\frac{1}{3}+0\right),\end{cases} \tag{19}$$

and for $\hat{u}=D(L)$

$$L\hat{u}=\left(-u'';u'\left(\frac{1}{3}-0\right)-u'\left(\frac{1}{3}+0\right),\hat{u}\in D(L)\right). \tag{20}$$

Lemma 2. *Operator defined by the formula (19), (20) is a linear closed operator with dense definitional domain in* $L_p(0,1)\oplus C.$ *Eigenvalues of the operator* L *and problem (1), (2) coincide, and* $\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ *are eigenvectors of the operator* L*, where* $Z^+=N\cup\{0\},$

$$\hat{u}_{i,n}=\left(u_{i,n}(x);mu_{i,n}\left(\frac{1}{3}\right)\right),\ i=1,2\quad n\in Z^+.$$

Proof. To prove the first part of the lemma we take $\hat{u}(u,\alpha)\in L_p(0,1)\oplus C$ and we define the functional $F(\hat{u})$ as follows:

$$F(\hat{u})=mu\left(\frac{1}{3}\right)-\alpha.$$

Let us assume

$$U_\nu(\hat{u})=U_\nu(u),\ \nu=1,2,3.$$

Then $F,U_\nu,\nu=1,2,3$ are bounded linear functionals on $W_p^k\left(0,\frac{1}{3}\right)\cup W_p^k\left(\frac{1}{3},1\right)\oplus C$, but unbounded on $L_p(0,1)\oplus C.$ Therefore (see, for example, [16, pp. 27-29]) the set

$$D(L)=\left\{\hat{u}=(u,\alpha),\, u\in W_p^2\left(0,\frac{1}{3}\right)\cup W_p^2\left(\frac{1}{3},1\right), F(\hat{u})=U_\nu(\hat{u})=0, \nu=1,2,3\right\}$$

is everywhere dense in $L_p(0,1)\oplus C,$ and L is a closed operator as contraction of corresponding closed maximal operator.

The second part of the lemma is verified directly. The lemma is proved.

For construction resolvent of operator L, consider the equation

$$L\hat{u}-\lambda\hat{u}=\hat{f}, \tag{21}$$

where $\hat{u}\in D(L),\ \hat{f}=(f,\beta)\in L_p(0,1)\oplus C.$ We can rewrite equation (22) in the form of

$$\begin{cases} -u''=\lambda u+f, \\ u'\left(-\frac{1}{3}\right)-u'\left(\frac{1}{3}\right)-\lambda mu\left(\frac{1}{3}\right)=\beta, \\ U_\nu(u)=0,\ \nu=1,2,3. \end{cases} \tag{22}$$

Lemma 3. *For solution* $\hat{u}=\left(u, mu\left(\tfrac{1}{3}\right)\right)$ *of the equation (22) it holds the following representations*

$$\begin{aligned} u(x,\rho)=&\frac{\beta\rho^2}{\Delta(\rho)}\left(e^{\frac{2i\rho}{3}}[1]+e^{-\frac{2i\rho}{3}}[1]\right)\left(e^{i\rho x}[1]+e^{-i\rho x}[1]\right)+ \\ &+\int_0^{\frac{1}{3}}G_{11}(x,\xi,\rho)f(\xi)d\xi+\int_{\frac{1}{3}}^1 G_{12}(x,\xi,\rho)f(\xi)d\xi,\quad x\in\left[0,\frac{1}{3}\right], \end{aligned} \tag{23}$$

$$\begin{aligned} u(x,\rho)=&\frac{\beta\rho^2}{\Delta(\rho)}\left(e^{\frac{i\rho}{3}}[1]+e^{-\frac{i\rho}{3}}[1]\right)\left(e^{i\rho(x-1)}[1]+e^{-i\rho(x-1)}[1]\right)+ \\ &+\int_0^{\frac{1}{3}}G_{21}(x,\xi,\rho)f(\xi)d\xi+\int_{\frac{1}{3}}^1 G_{22}(x,\xi,\rho)f(\xi)d\xi,\quad x\in\left[\frac{1}{3},1\right], \end{aligned} \tag{24}$$

$$u\left(\frac{1}{3},\rho\right)=\frac{\beta\rho^2}{\Delta(\rho)}\left(e^{\frac{2i\rho}{3}}[1]+e^{-\frac{2i\rho}{3}}[1]\right)\left(e^{\frac{i\rho}{3}}[1]+e^{-\frac{i\rho}{3}}[1]\right)+$$

$$+\int_0^{\frac{1}{3}}G_{21}(x,\xi,\rho)f(\xi)d\xi+\int_{\frac{1}{3}}^{1}G_{22}(x,\xi,\rho)f(\xi)d\xi,\ \ x\in\left[0,\frac{1}{3}\right]. \tag{25}$$

Proof. The solution of (22) will be sought in the form

$$u(x,\rho)=\begin{cases} c_{11}y_{11}(x)+c_{12}y_{12}(x)+y_1(x), & x\in\left[0,\frac{1}{3}\right],\\ c_{21}y_{21}(x)+c_{22}y_{22}(x)+y_2(x), & x\in\left[\frac{1}{3},1\right],\end{cases} \tag{26}$$

where $y_1(x)$ and $y_2(x)$ are defined by (13). Since $y(x)$, defined by (9) satisfies boundary conditions (2), then

$$U_\nu(u)=0,\ \nu=1,2,3. \tag{27}$$

Let's demand the function $u(x,\rho)$ satisfy boundary conditions $U_\nu(u)=0,\ \nu=1,2,3;\ U_4(u)=\beta$. Then taking into account (27) from (26) we obtain

$$\begin{cases} C_{11}U_{\nu1}(y_{11})+C_{12}U_{\nu1}(y_{12})+C_{21}U_{\nu2}(y_{21})+C_{22}U_{\nu2}(y_{22}),\nu=1,2,3;\\ C_{11}U_{41}(y_{11})+C_{12}U_{41}(y_{12})+C_{21}U_{42}(y_{21})+C_{22}U_{42}(y_{22})=\beta.\end{cases}$$

Solving this system with respect to unknowns $C_{j,k}$, we get

$$C_{11}=-\frac{\beta}{\Delta(\rho)}U_{11}(y_{12})[U_{22}(y_{21})U_{32}(y_{22})-U_{22}(y_{22})U_{32}(y_{21})],$$

$$C_{12}=\frac{\beta}{\Delta(\rho)}U_{11}(y_{11})[U_{22}(y_{21})U_{32}(y_{22})-U_{22}(y_{22})U_{32}(y_{21})],$$

$$C_{21}=\frac{\beta}{\Delta(\rho)}U_{22}(y_{22})[U_{11}(y_{11})U_{31}(y_{12})-U_{11}(y_{12})U_{31}(y_{11})],$$

$$C_{22}=-\frac{\beta}{\Delta(\rho)}U_{22}(y_{21})[U_{11}(y_{11})U_{31}(y_{12})-U_{11}(y_{12})U_{31}(y_{11})].$$

Substituting the received values of the coefficients $C_{j,k}$ in (26) and taking into account formula (3) ,(15)-(18) , we obtain the validity of formulas (23) and (24) . And formula (25) is obtained from (24) (or from (23)) by substitution $x=\frac{1}{3}$.
Lemma is proved.

4. Completeness of the eigenfunctions in spaces $L_p(0,1)\oplus C$ and $L_p(0,1)$

***Theorem 3**. System $\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ of eigenvectors of the operator L is complete in $L_p(0,1)\oplus C, 1< p<\infty.$*

Proof. To prove the completeness of the system of eigenfunctions of the operator L in $L_p(0,1)\oplus C$ we need to get the estimation of the resolvent of the operator L at great values of $|\rho|$. Assume $G(\delta)=C\setminus\bigcup\limits_{j,n}K_{j,n}(\delta)$. From the representations (24), (25) considering the inequalities (5) –(7) we obtain the inequality

$$|u(x,\rho)|\le\frac{C_\delta}{|\rho|}\|f\|_{L_p},\ \rho\in G(\delta),\quad |\rho|\ge r_0,$$

which fairly uniform on $x\in[0,1]$. From the last estimation it follows that for the resolvent $R(\lambda)=(L-\lambda I)^{-1}$ of the operator L outside of the above-stated circles the following estimation is true

$$\|R(\rho^2)\|\le\frac{C_\delta}{|\rho|},\ |\rho|\ge r_0. \tag{28}$$

Having estimation (28), by a standard method (see for example [17]) we obtain that eigenfunctions of operator L form a complete system in $L_p(0,1)\oplus C$.

Let us note that the system $\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ of eigenvectors of the operator L has a biorthogonal-conjugate system $\{\hat{v}_{i,n}\}_{i=1,2;n\in Z^+}$, where $\hat{v}_{i,n}=\left(v_{i,n}(x);\overline{m}v_{i,n}\left(\frac{1}{3}\right)\right)$ that are the system of eigenvectors of the conjugate operator L^*, which in its turn is the linearized operator of the conjugate spectral problem:

$$-v''+\overline{q(x)}v=\lambda v,\quad x\in\left(0;\frac{1}{3}\right)\cup\left(\frac{1}{3},1\right), \tag{1^*}$$

$$\left.\begin{array}{l} v'(0) = v'(1) = 0, \\ v\left(\frac{1}{3} - 0\right) = v\left(\frac{1}{3} + 0\right), \\ v'\left(\frac{1}{3} - 0\right) - v'\left(\frac{1}{3} + 0\right) = \lambda m v\left(\frac{1}{3}\right). \end{array}\right\} \quad (2^*)$$

Taking into account this by Theorem 3 we obtain

Corollary. *System* $\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^+}$ *of the eigenvectors of the operator* L *is complete and minimal in* $L_p(0,1)\oplus C, 1 < p < \infty.$

Now let us consider the completeness and minimality of a system $\{u_{i,n}\}_{i=1,2;n\in Z^+}$ of eigenfunctions of the problem (1), (2). It is clear that this system is overflowing in $L_p(0,1)$ one function of this system is unnecessary. Let us clarify the following question: Which function can be excluded from the system while maintaining the properties of completeness and minimality and which can not? The answer to this question is given by the following theorem.

Theorem 4. *Let* n_0 *be some number from the set of indexes* Z^+. *Then the system obtained from the system* $\{y_{i,n}\}_{i=1,2;n\in Z^+}$ *by excluding an arbitrary function* y_{i,n_0}, *where for* n_0 *condition* $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0$ *is satisfied, is complete and minimal in the spac* $L_p(0,1), 1 < p < \infty.$

Proof. According to the above mentioned, the system $\{\hat{u}_{i,n}\}_{i=1,2;n\in Z^+}$ has a

biortogonal- conjugated system $\{\hat{v}_{i,n}\}_{i=1,2;n\in Z^+}$, where $\hat{v}_{i,n} = \left(v_{i,n}(x); \overline{m} v_{i,n}\left(\frac{1}{3}\right)\right),$

And $v_{i,n_0}(x)$ are eigenfunctions of the adjoint problem $(1^*), (2^*)$. According to this formula, if condition $v_{i,n_0}\left(\frac{1}{3}\right) \neq 0$ is satisfied for some $n_0 \in Z^+$, then all conditions of theorem from [17] (see also [18]) are satisfied, from which the validity of the statement of our theorem is obtained.

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References

[1] Tikhonov AN, Samarskii AA, *Equations of Mathematical Physics*. Moscow: Nauka; 1977.

[2] Atkinson FV, *Discrete and Continuous Boundary Problems*. Moscow: Mir; 1968

[3] Collatz L, *Eigenvalue problems*. Moscow: Nauka; 1968.

[4] Gasymov TB, Mammadova SJ. On convergence of spectral expansions for one

[5] discontinuous problem with spectral parameter in the boundary condition. *Trans. of NAS of Azerb* 2006, v. 26 (4), p. 103-116.

[6] Gasymov TB, Huseynli AA. The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition, *Proc. of IMM of NAS of Azerb* 2011, v. 35 (18), p. 21-32.

[7] Gasymov TB, Maharramova GV, Mammadova NG. Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces. *Transactions of NAS of Azerbaijan, Issue Mathematics* 2018, v. 38 (1), p. 62-68.

[8] Gasymov TB, Maharramova GV, Jabrailova AN. Spectral properties of the problem of vibration of a loaded string in Morrey type spaces. *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan* 2018, v. 44 (1), p. 116-122.

[9] Bilalov BT, Gasymov TB, Maharramova GV. On basicity of eigenfunctions of one discontinuous spectral problem in Morrey type spaces, *The Aligarh Bulletin of mathematics 35:(1-2),2016,119-129*.

[10] Gasymov TB, Akhtyamov AM, Ahmedzade NR. On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces. *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan* 2020, v. 46 (1), p. 32–44.

[11] Bilalov BT, Gasymov TB, Maharramova GV. On basicity of eigenfunctions of spectral problem with discontinuity point in Lebesgue spaces, *Diff. equation* 2019, v. 55 (12), p. 1544-1553.

[12] Akhmedov AQ, Feyzullayev IQ. On completeness of Eigenfunctions of the Spectral Problem. *Caspian Journal of Applied Mathematics, Ecology and*

Economics 2022, v. 10 (2), p. 43-54.

[13] Gasymov TB, Akhmedov AQ. On basicity of eigenfunctions of a spectral problem in $L_p \oplus C$ and L_p spaces, *Baku State University Journal of Mathematics* & Computer Sciences 2024, v.1(1), p. 37-51.

[14] Gasymov TB, Akhmedov AQ, Taghiyeva R.J. On Basicity of Eigenfunctions of One Spectral Problem with the Discontinuity Point in Morrey-Lebesgue Spaces, *Caspian Journal of Applied Mathematics, Ecology and Economics* 2023, v. 12 (2), p. 42-52.

[15] Ahmadov AQ. Asymptotics of the Eigenvalues and Eigenfunctions of a Differential Operator with a conjugation conditions and a Summable Potential, *Caspian Journal of Applied Mathematics, Ecology and Economics* 2024, v. 12 (2), p. 30-41.

[16] Naimark MA. Linear Differential Operators, *2nd ed. Ungar, New York, 1967.*

[17] Lyantse VE, Storozh OG. Methods of the theory of unbounded operators, *Kiev, 1983.*

[18] Dunford, Schwartz JT. Linear operators, part3, *New York, Sydney, Toronto, 1971.*

[19] Gasymov TB. On necessary and sufficient conditions of basicity of some defective system in Banach spaces. *Trans. NAS Azerb.* 2006, v. 26 (1), p. 65-70.

[20] Gasymov TB, Garayev TZ. On defective bases from the root elements of differential operators containing a spectral parameter in the boundary conditions, *Trans. NAS Azerb.* 2007, v. 27 (1), p. 51-54.