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CONSTRUCTING FUNDAMENTAL SOLUTION AND THE SYSTEM OF INDEPENDENT PARTICULAR SOLUTIONS FOR A COMPLEX PARAMETER-DEPENDENT SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

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Abstract

In the present paper, in some part of the λ -plane we construct a fundamental solution for a complex parameter-dependent linear ordinary differential equation with variable coefficients. Using the fundamental solution, some systems of independent particular solutions of the equation under consideration, are obtained. The studied equation includes parametric equations corresponding to parabolic equations in the sense of I.G.Petrovsky.

Keywords: parametric equation, fundamental solution, systems of independent particular solutions.

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1. Introduction

Owing to extensive empirical experience, traditional speech recognition methods, In some part of the λ plane we construct a fundamental solution and obtain some systems of independent particular solution of the equation

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$$L\left(x,\frac{d}{dx},\lambda^2\right)y\equiv\left(\sum_{k=0}^{2}A_k(x)\frac{d^k}{dx^k}-\lambda^2\right)y=0,\qquad x\in(0,1),\tag{1}$$

where $A_k(x)$ are known (complex-valued) functions, λ is a complex parameter.

1^0. Let $A_j(x)\in C^{j+m+n}([0,1])$ and $|A_2(x)|>0$ for $x\in[0,1]$, where $j=0,1,2$ and $m(m\geq 3)$, $n(n\geq 0)$ be some integers.

2^0. Let $|\arg A_2(x)|\leq\frac{\pi}{2}-4\alpha$ for $x\in[0,1]$, where $\alpha\left(0<\alpha\leq\frac{\pi}{8}\right)$ is some number.

Remark 1. If

$$A_2(x)=a(x)+\sqrt{-1}\,b(x),$$

where $a(x)$ and $b(x)$ are real functions, then parabolicity (in the sense of I.G.Petrovsky [1]) of the equation

$$L\left(x,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=0,\tag{2}$$

$$x\in(0,1),\quad t>0,$$

means that the inequality

$$a(x)>0 \text{ for } x\in[0,1]\tag{3}$$

is fulfilled.

The parametric equations [6-7] corresponding to (2), (3) are within the equations (1) and in [5] this equation is studied for the case

$$\frac{b(x)}{a(x)}=const, \text{ for } x\in[0,1].\tag{4}$$

One of the main distinctions of the present paper from [2-5] is that we do not impose a restriction on (4).

Assume

$$\theta(x)=\frac{1}{\sqrt{A_2(x)}},$$

$$R_\alpha = \left\{ \lambda : |\lambda| \ge R,\ |\arg\lambda| \le \frac{\pi}{4} + \alpha \right\},$$

where R is any fixed positive number satisfying the below given inequality (12).

2. Solution

Following [2-7], we look formally for the particular solution $z_j(x,\lambda)$ of the equation (1) in the form

$$z_j(x,\lambda) = \exp(-\lambda\theta_j(x)) \sum_{\nu=0}^{\infty} \frac{1}{\lambda^\nu} g_{j\nu}(x),\ j = 1,2, \tag{5}$$

and as a result we obtain

$$\theta_1(x) = \int_0^x \theta(\xi)d\xi; \quad \theta_2(x) = \int_x^1 \theta(\xi)d\xi;$$

$$g_{j0}(x) = \frac{1}{\sqrt{\theta(x)}} \exp\left(-\frac{1}{2} \int_0^x \frac{A_1(\xi)}{A_2(\xi)} d\xi \right),$$

$g_{j\nu}(x), (\nu \ge 1)$ are some functions (see [2-7]) expressed by the recurrent formulas through the coefficients (and their derivatives) of the equation (1).

Remark 2. In [2-7] it is proved that for all $\nu = 0,1,2,\ldots,$ for determining $g_{j\nu}(x)$ we need infinite differentiability of the coefficients $A_k(x), k = 0,1,2$. In what follows, the results of the papers [6-7] yield that under the constraints 1^0 all the functions $g_{j\nu}(x)$ for $\nu = 0,1,2,\ldots,m$ are determined, and the following inclusion

$$\theta(x),\ g_{j\nu}(x) \in C^{2+n}([0,1]) \ \text{ for } \ \nu = 0,1,\ldots,m.$$

holds.

Assume

$$y_j^0(x,\lambda)=\exp\left(-\lambda\theta_j(x)\right)\sum_{\nu=0}^{m}\frac{1}{\lambda^{\nu}}g_{j\nu}(x),\ \ j=1,2. \tag{6}$$

From the above stated ones we have the following theorem.

Theorem 1. Under the constaints 1^0 for the functions $y_j^0(x,\lambda)$ determined by formula (6) we have

$$L\left(x,\frac{d}{dx},\lambda^2\right)y_j^0(x,\lambda)=\frac{1}{\lambda^m}q_{jm}(x)\exp\left(-\lambda\theta_j(x)\right),\ \ j=1,2, \tag{7}$$

where $q_{jm}(x)$ are some functions satisfying the inclusions

$$q_{jm}(x)\in C^n([0,1]),\ \ j=1,2.$$

To construct the fundamental solution $P(x,\xi,\lambda)$ of the equation (1) according to the Elmaga technigue [6-7], its principal part $P_0(x,\xi,\lambda)$ is determined as follows:

$$P_0(x,\xi,\lambda)=-\frac{1}{A_2(\xi)W_0(\xi,\lambda)}\cdot\begin{cases}y_1^0(x,\lambda)y_2^0(\xi,\lambda), & for\ 0\le\xi\le x\le1,\\ y_2^0(x,\lambda)y_1^0(\xi,\lambda), & for\ 0\le x\le\xi\le1,\end{cases} \tag{8}$$

where

$$W_0(\xi,\lambda)=y_1^0(\xi,\lambda)\frac{d}{d\xi}y_2^0(\xi,\lambda)-y_2^0(\xi,\lambda)\frac{d}{d\xi}y_1^0(\xi,\lambda).$$

Remark 3. Here picking up the principal part $P_0(x,\xi,\lambda)$ by formula (8) is the second distinction of this paper from [2-5].

We note some properties of the function $P_0(x,\xi,\lambda)$:

1. $P_0(x,\xi,\lambda)$ is continuous for $0\le x\le1,\ 0\le\xi\le1$.
2. $\frac{\partial}{\partial x}P_0(x,\xi,\lambda)$ is continuous for $0\le x<\xi\le1,\ \ 0\le\xi<x\le1$ and

   $$\frac{\partial}{\partial x}P_0(x,\xi,\lambda)\Big/_{x=\xi+0}-\frac{\partial}{\partial x}P_0(x,\xi,\lambda)\Big/_{x=\xi-0}=\frac{1}{A_2(\xi)},\ \text{ for }\ 0<\xi<1.$$

3. $\frac{\partial^2}{\partial x^2}P_0(x,\xi,\lambda)$ is continuous for $0\le x<\xi\le1$, and $0\le\xi<x\le1$.

4. For any $\psi(x)\in C([0,1])$ for $0<x<1$ we have the equality

$$\frac{\partial}{\partial x}\int_0^1 P_0(x,\xi,\lambda)\psi(\xi)d\xi=\int_0^1\frac{\partial}{\partial x}P_0(x,\xi,\lambda)\psi(\xi)d\xi,$$

$$\frac{\partial^2}{\partial x^2}\int_0^1 P_0(x,\xi,\lambda)\psi(\xi)d\xi=\frac{1}{A_2(x)}\psi(x)+\int_0^1\frac{\partial^2}{\partial x^2}P_0(x,\xi,\lambda)\psi(\xi)d\xi .$$

Now, we look for the fundamental solution (see [5-7]) of the equation (1) in the form:

$$P(x,\xi,\lambda)=P_0(x,\xi,\lambda)+P_1(x,\xi,\lambda),$$

$$P_1(x,\xi,\lambda)=\int_0^1 P_0(x,\eta,\lambda)h(\eta,\xi,\lambda)d\eta, \tag{9}$$

where $h(x,\xi,\lambda)$ is a desired kernel.

From the properties of defining the fundamental solution it follows that $P(x,\xi,\lambda)$ should satisfy the equation

$$L\left(x,\frac{\partial}{\partial x},\lambda^2\right)P(x,\xi,\lambda)=0 \quad \text{for} \quad x\neq\xi,\ 0<\xi<1.$$

Using the above stated, for determining the kernel $h(x,\xi,\lambda)$ we obtain the following Fredholm second order integral equation,

$$h(x,\xi,\lambda)=K(x,\xi,\lambda)+\int_0^1 K(x,\eta,\lambda)h(\eta,\xi,\lambda)d\eta, \tag{10}$$

where $K(x,\xi,\lambda)=-L\left(x,\frac{\partial}{\partial x},\lambda^2\right)P_0(x,\xi,\lambda)$.

Now, we look (7) and (8) we have

$$|K(x,\xi,\lambda)|\leq\frac{B}{|\lambda|^{m-1}}\exp(-\varepsilon|\lambda||x-\xi|),$$

$$0\leq x,\ \xi\leq 1,\ \ \lambda\in R_\alpha, \tag{11}$$

where B and ε are some positive numbers independent of x,ξ and λ.

The estimation (11) shows that for

$$R > B^{\frac{1}{m-2}}. \qquad (12)$$

the equation (10) is a Fredholm second order integral equation and to this equation we can apply the principle of contracted mappings, and Neuman;s successive approximations uniformly converge to some limit function $h(x,\xi,\lambda)$ being the solution of the equation (10).

Thus, we established the following theorem:

Theorem 2. Under constraints 1^0 and 2^0 and $\lambda \in R_\alpha$ the equation (1) has a fundamental solution $P(x,\xi,\lambda)$ determined by (9) and for the remainder addend $P_1(x,\xi,\lambda)$ we have the estimations

$$\left|\frac{\partial^k}{\partial x^k} P_1(x,\xi,\lambda)\right| \le \frac{C}{|\lambda|^{m-k}} \exp\left(-\varepsilon|\lambda||x-\xi|\right),$$

$$x,\xi \in [0,1],\ \lambda \in R_\alpha,\ k = 0,1,2,$$

(here and further by C and ε we will denote various positive numbers whose specific values are not significant) and for any $\psi(x) \in C([0,1])$ we have the identity

$$L\left(x, \frac{d}{dx}, \lambda^2\right)\int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi = \psi(x), \quad 0 < x < 1. \qquad (13)$$

From (13) for $\psi(x) \in C^2([0,1])$ we have the identity

$$L\left(x, \frac{d}{dx}, \lambda^2\right)\psi(x) = L\left(x, \frac{d}{dx}, \lambda^2\right)\int_0^1 P(x,\xi,\lambda) L\left(\xi, \frac{d}{d\xi}, \lambda^2\right)\psi(\xi)d\xi .$$

Hence we obtain the following identity

$$L\left(x, \frac{d}{dx}, \lambda^2\right)\left\{\psi(x) - \int_0^1 P(x,\xi,\lambda) L\left(\xi, \frac{d}{d\xi}, \lambda^2\right)\psi(\xi)d\xi\right\} \equiv 0 .$$

The validity of the following theorem follows from the above.

Theorem 3. Under the constraints 1^0 and 2^0, for any $\psi(x) \in C^2([0,1])$ the function $y(x,\lambda)$ determined by the formula

$$y(x,\lambda)=\psi(x)-\int_0^1 P(x,\xi,\lambda)L\left(\xi,\frac{d}{d\xi},\lambda^2\right)\psi(\xi)d\xi \tag{14}$$

is the solution of the homogeneous equation (1).

In (14) assuming $\psi(x)=y_j^0(x,\lambda)$ and $y(x,\lambda)=y_j(x,\lambda)$ using (7) we have

$$y_j(x,\lambda)=y_j^0(x,\lambda)+\rho_j(x,\lambda),\quad (j=1,2),$$

$$\rho_j(x,\lambda)=-\int_0^1 P(x,\xi,\lambda)\frac{1}{\lambda^m}q_{jm}(\xi)\exp(-\lambda\theta_j(\xi))d\xi\,. \tag{15}$$

Thus we established the following theorem.

Theorem 4. Under constraints 1^0 and 2^0 and $\lambda\in R_\alpha$ the equation (1) has a system of independent particular solutions of $y_j(x,\lambda)$, $(j=1,2)$ determined by the formula (15) and for the remainder addend $\rho_j(x,\lambda)$ we have the estimations

$$\left|\frac{d^k}{dx^k}\rho_1(x,\lambda)\right|\le\frac{C}{|\lambda|^{m+1-k}}\exp(-\varepsilon|\lambda|x),$$

$$\left|\frac{d^k}{dx^k}\rho_2(x,\lambda)\right|\le\frac{C}{|\lambda|^{m+1-k}}\exp(-\varepsilon|\lambda|(1-x)),$$

$$x\in[0,1],\ \lambda\in R_\alpha,\ k=0,1,2\,.$$

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