Baku State University Journal of Mathematics & Computer Sciences ***2024****, v. 1 (2), p. 1-11*

journal homepage: http://bsuj.bsu.edu.az/en

Nodal solutions of nonlinear Sturm-Liouville problems with a spectral parameter in the boundary condition

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Received 10 January 2024; accepted April 15 April 2024

Abstract

This paper is devoted to the study of a nonlinear boundary value problem for the Sturm-Liouville equation with a parameter contained both in the equation and in the boundary condition. We show the existence of solutions to this problem with a fixed number of simple nodal zeros.

Keywords: nonlinear Sturm-Liouville problem, spectral parameter, simple nodal zero, bifurcation point, global bifurcation

Mathematics Subject Classification *(2020): 34A30, 34B09, 34B15, 34C23, , 47J10, 47J15*

1. Introduction

Let $h(t)$ is a continuous function on R that satisfies the following conditions:

(H_1) there exists positive constant h_0 such that

$$\lim_{|t| \to 0} \frac{h(t)}{t} = h_0,$$

(H_2) there exists positive constant h_∞ such that

$$\lim_{|t| \to \infty} \frac{h(t)}{t} = h_\infty.$$

In this paper we consider the following nonlinear problem

$$\ell(u)(x) \equiv (p(x)u'(x))' - q(x)u(x) = \chi r(x)h(u(x)),\ x \in (0,1), \tag{1}$$

$$a_0 u(0) - d_0 p(0)u'(0) = 0, \tag{2}$$

$$(a_1 \chi h_0 + b_1)u(1) - (c_1 \chi h_0 + d_1)p(1)u'(1) = 0, \tag{3}$$

where $p(x)$ is a positive continuously differentiable function on $[0, 1]$, $q(x)$ is a nonnegative continuous function on $[0, 1]$, $r(x)$ is a positive continuous function on $[0, 1]$, χ is a positive parameter, $\lambda \in R$ is a spectral parameter, b_0, d_0, a_1, b_1, c_1 and d_1 are constants such that

$$|b_0| + |d_0| > 0,\ \sigma_1 = a_1 d_1 - b_1 c_1 > 0. \tag{4}$$

It is known that nonlinear Sturm-Liouville problems play an important role in modern mathematics and physics. Such problems arise when studying various processes of mechanics and physics. For example, the problem (1)-(3) describes the problem of the forms of loss of stability of a rod, of unit length and variable stiffness, under longitudinal load or torsion, at one end of which the load is concentrated (see, e.g., [7, 8, 12, 15]).

Note that many papers have been published that are devoted to studying the existence of nodal solutions of nonlinear second-order differential equations under various boundary conditions (see, for example, [2, 4-6, 9, 11, 13, 14] and references therein). In these papers, using various methods, the authors established the existence of solutions with a fixed oscillation count of nonlinear boundary value problems for second-order ordinary differential equations. Unfortunately, the existence of nodal solutions to nonlinear boundary value problems for second-order ordinary differential equations with a parameter in the boundary conditions has not been studied.

The purpose of this paper is to find intervals of values of the parameter χ for which there are solutions to problem (1)-(3) with fixed oscillation count.

2. Preliminary

Below we will give some auxiliary statements from papers [2-4].

Let

$$E = C^1[0,1] \cap \{u : b_0 u(0) = d_0 p(0)u'(0)\}$$

be the Banach space with the norm

$$\|u\|_1 = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)|.$$

By S we denote the subset of E given as follows:

$$S=\{u\in E:|u(x)|+|u'(x)|>0,\ x\in[0,1]\}.$$

Let $\gamma(\lambda)$ be the continuous function on R defined by the formulas:

$$\cot\gamma(\lambda)=\frac{a_1\lambda+b_1}{c_1\lambda+d_1},\ \ \gamma\left(-\frac{d_1}{c_1}\right)=0. \tag{5}$$

By (4) it follows from (5) that the function $\gamma(\lambda)$ is a strictly decreasing on R. Next, for each $u\in S$, we define the continuous function $\theta(u,x)$ on $[0,1]$ as follows:

$$\cot\theta(u,x)=\frac{p(x)u'(x)}{u(x)},\ \ \theta(u,0)=\cot^{-1}\frac{b_0}{d_0}. \tag{6}$$

For each $k\in\mathrm{N}$, each $\nu\in\{+,-\}$ and each $\lambda\in R$ by $S^{\nu}_{k,\lambda}$ we denote the set of functions $u\in S$ which satisfy the following conditions:

(i) $\theta(u,1)=\gamma(\lambda)+(k-1)\pi$;

(ii) $\nu u(x)$ is positive in the punctured neighborhood of the point $x=0$.

For each fixed $\lambda\in R$ the sets $S^{+}_{k,\lambda},\ S^{-}_{k,\lambda}$ and $S_{k,\lambda}=S^{+}_{k,\lambda}\cup S^{+}_{k,\lambda},\ k\in\mathrm{N},\ \nu\in\{+,-\}$, are open subsets in the space E. Moreover, if $u\in\partial S^{\nu}_{k,\lambda}$, then the function u has at least one double zero in the interval $[0,1]$.

Remark 1. Let $u\in S_{k,\lambda}$ for some $k\in\mathrm{N}$ and $\lambda\in R$. It follows from (5), (6) and [4, Lemma 2.1] that if $\lambda<-d_1/c_1$, then u has exactly $k-1$ simple nodal zeros in $(0,1)$, and if $\lambda\geq -d_1/c_1$, then u has exactly $k-2$ simple nodal zeros in $(0,1)$.

For each $k\in\mathrm{N}$ and each $\nu\in\{+,-\}$ let

$$S^{\nu}_{k}=\bigcup_{\lambda\in R}S^{\nu}_{k,\lambda}\quad\text{and}\quad S_k=\bigcup_{\lambda\in R}S_{k,\lambda}.$$

By the above arguments the sets $S^{\nu}_{k}\ (S_k),\ k\in\mathrm{N}$, each $\nu\in\{+,-\}$, are open and disjoint subsets of the space E. Moreover, the boundaries of these sets contain functions that have at least one double zero in the interval $[0,1]$.

We consider the following linear eigenvalue problem

$$\begin{cases}\ell(u)(x)=\lambda r(x)u(x),\ x\in(0,1),\\ a_0u(0)-d_0p(0)u'(0)=0,\\ (a_1\lambda+b_1)u(1)-(c_1\lambda+d_1)p(1)u'(1)=0.\end{cases} \tag{7}$$

It follows from [4, Theorem 3.1] that the eigenvalues of problem are real and simple, and form an unboundedly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, for each $k\in\mathrm{N}$ the eigenfunction u_k corresponding to the eigenvalue λ_k satisfies the following relation:

$$\theta(u_k, 1) = \gamma(\lambda_k) + (k-1)\pi. \tag{8}$$

In view of (8), we get $u_k \in S_k$, and consequently, u_k has exactly $k-1$ simple nodal zeros in $(0, 1)$ if $\lambda_k < -d_1/c_1$, and has exactly $k-2$ simple nodal zeros in $(0, 1)$ if $\lambda_k \geq -d_1/c_1$.

Remark 2. We assume that the function $q(x)$ and the coefficients in the boundary conditions (2) and (3) are chosen so that the eigenvalues of the linear problem (7) are positive.

Alongside the nonlinear problem (1)-(3) we shall consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(u)(x) = \lambda r(x) u(x) + g(x, u(x), u'(x), \lambda), \ x \in (0, 1), \\ a_0 u(0) - d_0 p(0) u'(0) = 0, \\ (a_1\lambda + b_1) u(1) - (c_1\lambda + d_1) p(1) u'(1) = 0, \end{cases} \tag{9}$$

where the real-valued function g, continuous on $[0, 1] \times R^3$, satisfies the following condition: for every bounded interval $\Lambda \subset R$,

$$g(x, u, s, \lambda) = o(|u| + |s|) \ \text{ as } \ |u| + |s| \to 0, \tag{10}$$

and

$$g(x, u, s, \lambda) = o(|u| + |s|) \ \text{ as } \ |u| + |s| \to \infty, \tag{11}$$

uniformly for $(x, \lambda) \in [0, 1] \times \Lambda$.

Due to conditions (10) and (11), problem (9) is linearizable both at zero and also at infinity, and in both cases the corresponding linear problem is (7). Since all eigenvalues of problem (7) are real and simple, the points $(\lambda_k, 0)$, $k \in \mathrm{N}$, are the bifurcation points of problem (9) with respect to the line of trivial solutions, and the points (λ_k, ∞), $k \in \mathrm{N}$, are the asymptotic bifurcation points the same problem. Moreover, by [2, Theorem 2.2] and [3, Theorems 3.2 and 3.5] we have the following results.

Theorem 1. *For each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ there exists a continuum C_k^ν of nontrivial solutions to problem (9), which meets $(\lambda_k, 0)$, is contained in $R \times S_k^\nu$ and is unbounded in $R \times E$. Moreover, if C_k^ν meets $R \times \{\infty\}$ for some $\lambda \in R$, then $\lambda = \lambda_k$.*

Theorem 2. *For each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ there exists a continuum D_k^ν of nontrivial solutions to problem (9), which meets (λ_k, ∞) and is contained in $R \times S_k^\nu$.*

Moreover, either D_k^ν meets $R\times\{0\}$ for some $\lambda\in R$ (in this case $\lambda=\lambda_k$), or the projection of D_k^ν onto $R\times\{0\}$ is unbounded.

Theorem 3. *If C_k^ν meets $R\times\{\infty\}$ for some $\lambda\in R$, then $\lambda=\lambda_k$, and if D_k^ν meets $R\times\{0\}$ for some $\lambda\in R$, then $\lambda=\lambda_k$.*

3. Existence of solutions to problem (1)-(3) contained in classes $S_k^\nu, k\in \mathrm{N}, \nu\in\{+,-\}$

In this section we find the interval for the parameter χ for which there are solutions to problem (1)-(3) contained in the classes $S_k^\nu, k\in \mathrm{N}, \nu\in\{+,-\}$, i.e., we show the existence of nodal solutions to this problem.

Theorem 3. *Let conditions (4), (H₁), (H₂) be satisfied and for some $k\in\mathrm{N}$ the following condition holds:*

$$\frac{\lambda_k}{h_0}<\chi<\frac{\lambda_k}{h_\infty} \text{ or } \frac{\lambda_k}{h_\infty}<\chi<\frac{\lambda_k}{h_0}$$

Then for each $\nu\in\{+,-\}$ there is a solution ϑ_k^ν of problem (1)-(3) such that $\vartheta_k^\nu\in S_k^\nu$.

Proof. According to condition (H_1), the function can be represented in the following form:

$$h(t)=h_0t+\widetilde{h}(t), \tag{12}$$

where

$$\widetilde{h}(t)=o(|t|) \text{ as } |t|\to 0. \tag{13}$$

Then (1)-(3) takes the following form

$$\begin{cases} \ell(u)(x)=\chi h_0 r(x)u(x)+\chi r(x)\widetilde{h}(u(x)),\ x\in(0,1), \\ a_0u(0)-d_0p(0)u'(0)=0, \\ (a_1\chi h_0+b_1)u(1)-(c_1\chi h_0+d_1)p(1)u'(1)=0, \end{cases} \tag{14}$$

For the proof of this theorem we consider the following nonlinear eigenvalue problem

$$\begin{cases} (1/\chi h_0)\,\ell(u)(x)=\lambda r(x)u(x)+(1/h_0)\,r(x)\widetilde{h}(u(x)),\ x\in(0,1), \\ a_0u(0)-d_0p(0)u'(0)=0, \\ (a_1\lambda\chi h_0+b_1)u(1)-(c_1\lambda\chi h_0+d_1)p(1)u'(1)=0. \end{cases} \tag{15}$$

where $\lambda \in R$ is a spectral parameter.

Let

$$\tilde{g}(x,u,s,\lambda) = (1/h_0)\, r(x)\, \tilde{h}(u).$$

Then we have

$$\frac{|\,\tilde{g}(x,u,s,\lambda)|}{|u|+|s|} = \frac{(1/h_0)\,|\,r(x)\tilde{h}(u)|}{|u|+|s|} \le \frac{(1/h_0)r_1\,|\,\tilde{h}(u)|}{|u|},$$

where $r_1 = \max\limits_{x\in[0,1]} r(x),$ whence, by (13),

$$\tilde{g}(x,u,s,\lambda) = o(|u|+|s|) \text{ as } |u|+|s| \to 0, \tag{16}$$

uniformly for $(x,\lambda) \in [0,1] \times R.$ Hence by Theorem 1 for each $k \in \mathrm{N}$ and each $\nu \in \{+,-\}$ there exists a continuum $\tilde{C}_k^\nu$ of nontrivial solutions to problem (15), which meets $(\tilde{\lambda}_k, 0),$ is contained in $R \times S_k^\nu$ and is unbounded in $R \times E,$ where $\tilde{\lambda}_k$ is the k th eigenvalue of the linear problem

$$\begin{cases} \ell(u)(x) = \lambda\chi h_0 r(x) u(x),\ x \in (0,1), \\ a_0 u(0) - d_0 p(0) u'(0) = 0, \\ (a_1\lambda\chi h_0 + b_1) u(1) - (c_1\lambda\chi h_0 + d_1) p(1) u'(1) = 0. \end{cases} \tag{17}$$

It is obvious that

$$\lambda_k = \tilde{\lambda}_k \chi h_0,$$

which implies that

$$\tilde{\lambda}_k = \frac{\lambda_k}{\chi h_0}.$$

Thus for each $k \in \mathrm{N}$ and each $\nu \in \{+,-\}$ there exists a continuum $\tilde{C}_k^\nu$ of nontrivial solutions to problem (12), which meets $\left(\frac{\lambda_k}{\chi h_0}, 0\right),$ is contained in $R \times S_k^\nu$ and is unbounded in $R \times E.$

In the other hand, by (H_2) we get

$$h(t) = h_\infty t + \hat{h}(t), \tag{18}$$

where

$$\hat{h}(t) = o(|t|) \text{ as } |t| \to \infty. \tag{19}$$

In view of (12) and (18) we get

$$\tilde{h}(t) = (h_\infty - h_0)\, t + \hat{h}(t). \tag{20}$$

Consequently, (15) can be rewritten in the following form

$$\begin{cases} (1/\chi h_0)\,\ell(u) + (1 - h_\infty/h_0)\,r(x)u = \lambda\, r(x)u + (1/h_0)\,r(x)\,\hat{h}(u),\ x \in (0,1), \\ a_0 u(0) - d_0\, p(0)u'(0) = 0, \\ (a_1\lambda\chi h_0 + b_1)\,u(1) - (c_1\lambda\chi h_0 + d_1)\,p(1)\,u'(1) = 0. \end{cases} \tag{21}$$

By (19), for any sufficiently small fixed $\varepsilon > 0$ there is a sufficiently large $\Delta_\varepsilon > 0$ such that for any $t \in R,\ |t| > \Delta_\varepsilon$, the following relation holds:

$$|\hat{h}(t)| < \frac{h_0}{r_1}\varepsilon |t|. \tag{22}$$

Since $\hat{h}(t) \in C(R)$, there exists positive number M_ε such that

$$|\hat{h}(t)| \le \frac{h_0}{r_1} M_\varepsilon \text{ for any } t \in R,\ |t| \le \Delta_\varepsilon. \tag{23}$$

We choose $\hat{\Delta}_\varepsilon > \Delta_\varepsilon$ so large that $M_\varepsilon < \varepsilon\hat{\Delta}_\varepsilon$.

We introduce the notation:

$$\hat{g}(x,u,s,\lambda) = (1/h_0)\,r(x)\,\hat{h}(u).$$

Let $(u,s) \in R^2$ such that $|u| + |s| > \hat{\Delta}_\varepsilon$. Then by (22) and (23) we have

$$\frac{|\hat{g}(x,u,s,\lambda)|}{|u|+|s|} = \frac{(1/h_0)\,r(x)\,|\hat{h}(u)|}{|u|+|s|} < \frac{r_1}{h_0}\frac{|\hat{h}(u)|}{|u|} < \varepsilon \text{ for } |u| > \Delta_\varepsilon,$$

$$\frac{|\hat{g}(x,u,s,\lambda)|}{|u|+|s|} = \frac{(1/h_0)\,r(x)\,|\hat{h}(u)|}{|u|+|s|} < \frac{M_\varepsilon}{\hat{\Delta}_\varepsilon} = \varepsilon \text{ for } |u| \le \Delta_\varepsilon.$$

Thus

$$\tilde{g}(x,u,s,\lambda) = o(|u|+|s|) \text{ as } |u|+|s| \to \infty, \tag{24}$$

uniformly in $(x,\lambda) \in [0,1] \times R$, and consequently, problem (21) is asymptotically linear. Moreover, the linear problem

$$\begin{cases} (1/\chi h_0)\,\ell(u)(x) + (1 - h_\infty/h_0)\,r(x)u(x) = \lambda\, r(x)u(x),\ x \in (0,1), \\ a_0 u(0) - d_0\, p(0)u'(0) = 0, \\ (a_1\lambda\chi h_0 + b_1)\,u(1) - (c_1\lambda\chi h_0 + d_1)\,p(1)\,u'(1) = 0. \end{cases} \tag{25}$$

or

$$\begin{cases} \ell(u)(x) = (\lambda + h_\infty/h_0 - 1)\,h_0\chi\, r(x)\,u(x),\ x \in (0,1), \\ a_0 u(0) - d_0\, p(0)\,u'(0) = 0, \\ (a_1\lambda\chi h_0 + b_1)\,u(1) - (c_1\lambda\chi h_0 + d_1)\,p(1)\,u'(1) = 0. \end{cases} \tag{26}$$

is the corresponding linear problem. It is clear from (26) that

$$\lambda_k = (\hat{\lambda}_k + h_\infty / h_0 - 1) h_0 \chi,$$

where $\hat{\lambda}_k$ is the k th eigenvalue of problem (25), whence we get

$$\hat{\lambda}_k = \frac{\lambda_k}{\chi h_0} - \frac{h_\infty}{h_0} + 1.$$

By (24) it follows from Theorem 2 that for each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ there exists a continuum $\hat{D}_k^\nu$ of nontrivial solutions to problem (15) which meets $\left(\frac{\lambda_k}{\chi h_0} - \frac{h_\infty}{h_0} + 1, \infty\right)$ and is contained in $R \times S_k^\nu$. Moreover, either $\hat{D}_k^\nu$ meets $R \times \{0\}$ for some $\lambda \in R$ (in this case $\lambda = \tilde{\lambda}_k$), or the projection of $\hat{D}_k^\nu$ onto $R \times \{0\}$ is unbounded.

Now we will show that the projection of the set $\hat{D}_k^\nu$ onto $R \times \{0\}$ is bounded. Indeed, if the projection of the set $\hat{D}_k^\nu$ onto $R \times \{0\}$ is unbounded, then there exists the sequence $\{(\mu_n, \vartheta_n)\}_{n=1}^\infty \subset D_k^\nu \subset R \times S_k^\nu$ such that

$$\mu_n \to +\infty \quad \text{or} \quad \mu_n \to -\infty \quad \text{as} \quad n \to \infty. \tag{27}$$

By (20), it follows from (13) that for any sufficiently small fixed $\varepsilon_0 > 0$ there is a sufficiently small $\delta_0 > 0$ such that for any $t \in R,\ 0 < |t| < \delta_0$, the following relation holds:

$$\frac{|\hat{h}(t)|}{|t|} < |h_0 - h_\infty| + \frac{h_0}{r_1}\varepsilon_0. \tag{28}$$

By (22), for any $t \in R,\ |t| > \Delta_0 = \Delta_{\varepsilon_0}$ the following relation holds:

$$\frac{|\hat{h}(t)|}{|t|} < \frac{h_0}{r_1}\varepsilon_0. \tag{29}$$

Since $\hat{h}(t) \in C(R)$, there exists positive number K_0 such that

$$\frac{|\hat{h}(t)|}{|t|} \le K_0 \quad \text{for any } t \in R,\ \delta_0 \le |t| \le \Delta_0. \tag{30}$$

Then, it follows from (28)-(30) that

$$\frac{|\hat{h}(t)|}{|t|} \le N_0 \quad \text{for any } t \in R, t \ne 0, \tag{31}$$

where

$$N_0 = \max\left\{|h_0 - h_\infty| + \frac{h_0}{r_1}\varepsilon_0, K_0\right\}.$$

Let

$$\psi_n(x) = \begin{cases} -\dfrac{\tilde{h}(\vartheta_n(x))}{\vartheta_n(x)} & if \quad \vartheta_n(x) \neq 0, \\ 0 & if \ \ \vartheta_n(x) = 0. \end{cases} \tag{32}$$

By (31) it follows from (32) that

$$|\psi_n(x)| \leq N_0, \ x \in [0, 1]. \tag{33}$$

Since $\vartheta_n \in S_k^\nu$ for each $n \in \mathrm{N}$ it follows from (8) and [4, Theorem 3.1] that μ_n, $n \in \mathrm{N}$, is the k th eigenvalue of the linear spectral problem

$$\begin{cases} (1/\chi h_0 r(x))\,\ell(u)(x) + (1 - h_\infty/h_0)\ u(x) + (1/h_0)\psi_n(x) u(x) = \lambda u(x), \ x \in (0, 1), \\ a_0 u(0) - d_0 p(0) u'(0) = 0, \\ (a_1 \lambda \chi h_0 + b_1) u(1) - (c_1 \lambda \chi h_0 + d_1) p(1) u'(1) = 0. \end{cases} \tag{34}$$

Using operator interpretation of problem (34) [7, p. 295] by following the arguments in Lemma 4.2 of [1] we can show that

$$\lambda_{k,\chi} - \frac{N_0}{h_0} \leq \mu_n \leq \lambda_{k,\chi} + \frac{N_0}{h_0},$$

where $\lambda_{k,\chi}$ is the k th eigenvalue of the linear problem

$$\begin{cases} (1/\chi h_0 r(x))\,\ell(u)(x) + (1 - h_\infty/h_0)\ u(x) = \lambda u(x), \ x \in (0, 1), \\ a_0 u(0) - d_0 p(0) u'(0) = 0, \\ (a_1 \lambda \chi h_0 + b_1) u(1) - (c_1 \lambda \chi h_0 + d_1) p(1) u'(1) = 0. \end{cases}$$

which contradicts relation (27). Therefore, $\hat{D}_k^\nu$ meets $R \times \{0\}$ at the point $(\tilde{\lambda}_k, 0)$. In a similar way we can show that $\tilde{C}_k^\nu$ meets $R \times \{\infty\}$ at the point $(\hat{\lambda}_k, \infty)$. Consequently, for each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ the following relation holds:

$$\tilde{C}_k^\nu = \hat{D}_k^\nu \subset R \times S_k^\nu. \tag{35}$$

Assume that

$$\frac{\lambda_k}{h_0} < \chi < \frac{\lambda_k}{h_\infty} \text{ for some } k \in \mathrm{N}. \tag{36}$$

Then, by Remark 2, from the left side of the second relation in (36), we obtain

$$\tilde{\lambda}_k = \frac{\lambda_k}{h_0 \chi} < 1. \tag{37}$$

Moreover, taking into account (37) and the right-hand side of the second relation in (36), we get

$$1 < \frac{\lambda_k}{h_\infty \chi} = \frac{\lambda_k}{h_0 \chi}\frac{h_\infty}{h_0} = \frac{\lambda_k}{h_0 \chi} + \frac{\lambda_k}{h_0 \chi}\left(\frac{h_\infty}{h_0} - 1\right) < \frac{\lambda_k}{h_0 \chi} + \frac{h_\infty}{h_0} - 1 = \hat{\lambda}_k. \tag{38}$$

Since $\tilde{C}_k^\nu$ meets both points $(\tilde{\lambda}_k, 0)$ and $(\hat{\lambda}_k, \infty)$ and is connected, it follows from (37) and (38) that for each $\nu \in \{+,-\}$ there exists $\vartheta_k^\nu \in E$ such that

$$(1,\ \vartheta_k^\nu) \in \tilde{C}_k^\nu \subset R \times S_k^\nu. \tag{39}$$

By (39), it follows from (15) that ϑ_k^ν is a solution to problem (14), and therefore to problem (1)-(3), lies in S_k^ν.

The remaining cases are treated similarly. The proof of the theorem is complete.

References

[1] Aliyev ZS. Some global results for nonlinear fourth order eigenvalue problems, Cent. Eur. J. Math. **2014,** v. 12 (12), p. 1811-1828.

[2] Aliyev ZS, Ismayilova NA, Global bifurcation from zero in nonlinear Sturm-Liouville equation with a spectral parameter in the boundary condition. Quaest. Math. **2023,** v. 46 (11), 2233-2241.

[3] Aliyev ZS, Hadiyeva SS, Ismayilova NA. Global bifurcation from infinity in some nonlinear Sturm-Liouville problems. Bull. Malays. Math. Sci. Soc. **2023,** v. 46, p. 1-14.

[4] Binding PA, Browne PJ, Watson BA. Spectral problem for nonlinear Sturm-Liouville equations with eigenparameter dependent boundary conditions, Canad. J. Math. **2000,** v. 52 (2), p. 248-264.

[5] Binding PA, Browne PJ, Seddici K. Sturm-Liouville problems with eigen-parameter dependent boundary conditions. Proc. Edinb. Math. Soc. **1993,** 37 (1), p. 57-73.

[6] Dalbono F, Zanolin F. Multiplicity results for asymptotically linear equations, Using the rotation number approach. Mediterr. J. Math. **2007,** v. 4, p. 127-149.

[7] Fulton CT. Two-point boundary value problem with eigenvalue parameter contained in the boundary conditions. Proc. Roy. Soc. Edinburgh Sect. A **1977,** v. 77 (3-4), p. 293-308.

[8] Krasnoselskii MA, Zabreiko PP. Geometrical methods of nonlinear analysis, Springer-Verlag, Berlin, 1984.

[9] Ma R, Dai G. Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, J. Funct. Anal. **2013,** v. 265 (8), p. 1443-1459.

[10] Ma R, Thompson B. Nodal solutions for nonlinear eigenvalue problems, Nonlinear Anal. **2004,** v. 59 (5), p. 707-718.

[11] Naito Y, Tanaka S. On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, Nonlinear Anal. **2004,** 56 (4), p. 919-935.

[12] Roseau M. Vibrations in mechanical systems. Analytical methods and applications, Springer-Verlag, Berlin, 1987.

[13] Rynne BP. Half-eigenvalues of self adjoint, $2m$ th order differential operators and semilinear problems with jumping nonlinearities, Differential Integral Equations **2001,** v. 14 (9), p. 1129-1152.

[14] Tanaka S. On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem. Differential Integral Equations **2007,** v. 20 (1), p. 93-104.

[15] Tikhonov AN, Samarskii AA. Equations of mathematical physics, Dover Publ., New York, 1963.