

## AN INTEGRAL BOUNDARY VALUE PROBLEM FOR A LOADED THIRD-ORDER PSEUDOPARABOLIC EQUATION WITH SINGULAR COEFFICIENTS

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### Abstract

The paper investigates a boundary value problem with an integral boundary condition for a loaded third-order pseudoparabolic equation with singular coefficients. The solution of the problem is sought in a weighted Sobolev space. Using the integral representation of functions in this space, the formulated problem is transformed into an equivalent integral equation. The existence and uniqueness of the solution to both the integral equation and the original problem are established.

**Keywords:** pseudoparabolic equation, equation with singular coefficients, problem with integral boundary condition.

**Mathematics Subject Classification (2020):** 35A01, 35A02

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### 1. Introduction

Let us consider a third-order loaded pseudoparabolic equation with singular coefficients in the domain

$$\Omega = \left\{ t = (t_1, t_2) : t_i \in \Omega_i = (t_i^0, t_i^1), -\infty < t_i^0 < t_i^1 < +\infty, i = 1, 2 \right\};$$

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$$\begin{aligned}
 (I_{1,2}z)(t) \equiv & D_1 D_2^2 z(t) + \frac{e_{02}(t)}{(t_1 - t_1^0)^{\alpha_1}} D_2^2 z(t) + \frac{e_{11}(t)}{(t_2 - t_2^0)^{\alpha_2}} D_1 D_2 z(t) + \frac{e_{10}(t)}{(t_2 - t_2^0)^{\alpha_2}} D_1 z(t) + \\
 & + \frac{e_{01}(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}} D_2 z(t) + \frac{e_{00}(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}} z(t) + \quad (1) \\
 & + \frac{e(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}} z(\bar{t}) = \frac{\psi_{12}(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}}, \quad t \in \Omega
 \end{aligned}$$

Since the coefficients in equation (1) have singularities along the lines  $t_1 = t_1^0$ ,  $t_2 = t_2^0$ , these equations are referred to as equations with singular coefficients. The singularity lines also constitute the characteristics of the given equation. Note that in equation (1), when  $\alpha_1 = \alpha_2 = 0$  and  $e(t) \equiv 0, t \in \Omega$ , the obtained equation and its special cases have been studied within various local and nonlocal boundary conditions in [1-9] works. Such equations arise in the theory of filtration of fluids in porous media [10], in the unstable movement of free surface groundwater [11], in the theory of moisture transport in soil [12], and in many other fields. In these works, equation (1) is studied under various smoothness conditions imposed on its coefficients and on the right-hand side, and the solution is sought in the spaces  $C^{(1,2)}(\Omega)$  or  $W_{p,\alpha}^{(1,2)}(\Omega)$ . If  $\alpha_1^2 + \alpha_2^2 > 0$ , a solution of equation (1) may not exist in these spaces. Therefore, when  $\alpha_1^2 + \alpha_2^2 > 0$  it is necessary to introduce a space in which the solution of equation (1) exists. In the work [13], the weighted space  $W_{p,\alpha}^{(1,2)}(\Omega)$  was introduced, and the Cauchy problem for the unloaded equation (1) was solved.

In this work, equation (1) is investigated under integral boundary conditions. The solution to the posed problem is sought in the weighted Sobolev space  $W_{p,\alpha}^{(1,2)}(\Omega)$ . By using the integral representation of functions in this space, the posed problem is transformed to an equivalent integral equation. The existence and uniqueness of the solution to both the integral equation and the posed problem are proved.

## 2. Statement of the problem and its correctness

We consider equation (1) with nonlocal integral boundary condition

$$\begin{aligned}
 (l_{02}z)(t_2) &\equiv \beta_1 D_2^2 z(t_1^0, t_2) + \int_{t_1^0}^{t_1^1} \beta_2(t_1) D_2^2 z(t_1, t_2) dt_1 = \frac{\psi_{02}(t_2)}{(t_2 - t_2^0)^{\alpha_2}}, \quad t_2 \in \Omega_2, \\
 (l_{11}z)(t_1) &\equiv D_1 D_2 z(t_1, t_2^0) = \frac{\psi_{11}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, \quad t_1 \in \Omega_1, \\
 (l_{10}z)(t_1) &\equiv D_1 z(t_1, t_2^0) = \frac{\psi_{10}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, \quad t_1 \in \Omega_1,
 \end{aligned} \tag{2}$$

$$l_{01}z \equiv D_2 z(t_1^0, t_2^0) = \psi_{01}, \quad l_{00}z \equiv z(t_1^0, t_2^0) = \psi_{00},$$

where  $\Omega$  -is a given set,  $z(t)$ – is a sought function,  $e(t)$ ,  $e_{ij}(t)$ ,  $i=0,1$ ,  $j=0, 1, 2$ ,  $i + j < 3$  are measurable functions on the set  $\Omega$ ,

$e(t)$ ,  $e_{00}(t)$ ,  $e_{01}(t) \in L_p(\Omega)$ ,  $\psi_{12}(t) \in L_p(\Omega)$ ,  $\psi_{02}(t_2) \in L_p(\Omega_2)$ ,  $\psi_{1j}(t_1) \in L_p(\Omega_1)$  are given functions, there exist functions  $e_{02}^0(t_1) \in L_p(\Omega_1)$ ,

$e_{1j}^0(t_2) \in L_p(\Omega_2)$ ,  $j = 0,1$  such that the posed conditions hold for almost all  $t \in \Omega$ ,

$$|e_{02}(t)| \leq e_{02}^0(t_1) \in L_p(\Omega_1) \quad \text{and} \quad |e_{1j}(t)| \leq e_{1j}^0(t_2) \in L_p(\Omega_2), \quad j = 0,1.$$

$\psi_{0i} \in R, i=0,1$ ,  $1 < p < \infty$ ,  $\alpha_i \in \left[0,1 - \frac{1}{p}\right)$ ,  $i = 1,2$ ,  $\beta_1$  are given numbers,

$\bar{t} \in \Omega$ – is a given point,  $\beta_1 + \int_{t_1^0}^{t_1^1} \beta_2(t_1) dt_1 \neq 0$ . The solution of problem (1), (2) is

sought in the weighted Sobolev space

$$\begin{aligned}
 W_{p,\alpha}^{(1,2)}(\Omega) &= \left\{ z : D_1 D_2^2 z \in L_{p,\alpha}(\Omega), D_2^2 z \in L_{p,\alpha_2}(\Omega), D_1 D_2 z \in L_{p,\alpha_1}(\Omega), \right. \\
 &\quad \left. D_1 z \in L_{p,\alpha_1}(\Omega), D_2 z \in L_p(\Omega), z \in L_p(\Omega) \right\}.
 \end{aligned}$$

The norm in this space is defined as

$$\|z\|_{W_{p,\alpha}^{(1,2)}(\Omega)} = \|D_1 D_2^2 z\|_{p,\alpha,\Omega} + \|D_2^2 z\|_{p,\alpha_2,\Omega} + \|D_1 D_2 z\|_{p,\alpha_1,\Omega} + \|D_1 z\|_{p,\alpha_1,\Omega} + \|D_2 z\|_{p,\Omega} + \|z\|_{p,\Omega}.$$

It is known that [13] an arbitrary function  $z \in W_{p,\alpha}^{(1,2)}(\Omega)$  has the following unique representation:

$$z(t) = (Uv)(t) \equiv v_{00} + (t_2 - t_2^0)v_{01} + \int_{t_1^0}^{t_1} \frac{1}{(t_1 - t_1^0)^{\alpha_1}} (v_{10}(t_1) + (t_2 - t_2^0)v_{11}(t_1)) dt_1 + \int_{t_2^0}^{t_2} \frac{t_2 - \tau_2}{(t_2 - t_2^0)^{\alpha_2}} v_{02}(\tau_2) d\tau_2 + \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \frac{t_2 - \tau_2}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}} v_{12}(t_1, \tau_2) d\tau_1 d\tau_2, \tag{3}$$

here

$$v = \left( \frac{v_{12}(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}}, \frac{v_{02}(t_2)}{(t_2 - t_2^0)^{\alpha_2}}, \frac{v_{11}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, \frac{v_{10}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, v_{01}, v_{00} \right) \in H_{p,\alpha}^{(1,2)},$$

$$H_{p,\alpha}^{(1,2)} = L_{p,\alpha}(\Omega) \times L_{p,\alpha_2}(\Omega_2) \times L_{p,\alpha_1}(\Omega_1) \times L_{p,\alpha_1}(\Omega_1) \times R \times R, v_{12} \in L_p(\Omega), v_{02} \in L_p(\Omega_2),$$

$$v_{11}, v_{10} \in L_p(\Omega_1), v_{01}, v_{00} \in R.$$

We define the norm in the  $H_{p,\alpha}^{(1,2)}$  as follows:

$$\|v\|_{H_{p,\alpha}^{(1,2)}} = \left\| \frac{v_{12}}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}} \right\|_{p,\alpha,\Omega} + \left\| \frac{v_{02}}{(t_2 - t_2^0)^{\alpha_2}} \right\|_{p,\alpha_2,\Omega_2} + \left\| \frac{v_{11}}{(t_1 - t_1^0)^{\alpha_1}} \right\|_{p,\alpha_1,\Omega_1} + \left\| \frac{v_{10}}{(t_1 - t_1^0)^{\alpha_1}} \right\|_{p,\alpha_1,\Omega_1} + |v_{01}| + |v_{00}|.$$

By (3), it follows that any function  $z \in W_{p,\alpha}^{(1,2)}(\Omega)$  has traces

$$D_2^2 z(t_1^0, t_2), D_1 D_2^i z(t_1, t_2^0), i = 0, 1, D_2^i z(t_1^0, t_2^0), i = 0, 1$$

and these trace operators map continuously from the space  $W_{p,\alpha}^{(1,2)}(\Omega)$  into the spaces  $L_{p,\alpha_2}(\Omega_2)$ ,  $L_{p,\alpha_1}(\Omega_1)$  and  $R$  respectively. For these traces, the following equalities hold:

$$D_2^2 z(t_1^0, t_2) = \frac{v_{02}(t_2)}{(t_2 - t_2^0)^{\alpha_2}}, D_1 D_2^i z(t_1, t_2^0) = \frac{v_{1i}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, i = 0, 1,$$

$$D_2^i z(t_1^0, t_2^0) = v_{0i}, i = 0, 1. \tag{4}$$

Let's write problem (1), (2) in the form of an operator equation:

$$lz = \psi, \tag{5}$$

where  $l = (l_{12}, l_{02}, l_{11}, l_{10}, l_{01}, l_{00}): W_{p,\alpha}^{(1,2)}(\Omega) \rightarrow H_{p,\alpha}^{(1,2)}$ ,

$$\psi = \left( \frac{\psi_{12}(t)}{(t_1 - t_1^0)^{\alpha_1} (t_2 - t_2^0)^{\alpha_2}}, \frac{\psi_{02}(t_2)}{(t_2 - t_2^0)^{\alpha_2}}, \frac{\psi_{11}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, \frac{\psi_{10}(t_1)}{(t_1 - t_1^0)^{\alpha_1}}, \psi_{01}, \psi_{00} \right) \in H_{p,\alpha}^{(1,2)}.$$

Under the conditions imposed on the coefficients of problems (1), (2) the operator  $l : W_{p,\alpha}^{(1,2)}(\Omega) \rightarrow H_{p,\alpha}^{(1,2)}$  is linear and bounded. The following estimate holds for the operator  $Q$ :

$$k_1 \|v\|_{H_{p,\alpha}^{(1,2)}} \leq \|Uv\|_{W_{p,\alpha}^{(1,2)}(\Omega)} \leq k_2 \|v\|_{H_{p,\alpha}^{(1,2)}},$$

where  $k_1, k_2 > 0$  – are constants.

The operator  $U$  establishes an isomorphism between the spaces  $H_{p,\alpha}^{(1,2)}$  and  $W_{p,\alpha}^{(1,2)}$ . Hence, these spaces can be considered equivalent in the sense of isomorphism. Consequently, solving equation (5) is equivalent to solving equation

$$lUv = \psi. \tag{6}$$

(6) is equivalent to solving equation (6).

Now, using decomposition of the function  $z$  from (3), we rewrite equation (1) in the following form

$$\begin{aligned} (l_2 Uv)(t) &\equiv v_{12}(t) + e_{02}(t) \left[ v_{02}(t_2) + \int_{t_1^0}^{t_1} \frac{v_{12}(\tau_1, t_2)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 \right] + e_{11}(t) \left[ v_{11}(t_1) + \int_{t_2^0}^{t_2} \frac{v_{12}(t_1, \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} d\tau_2 \right] + \\ &+ e_{10}(t) \left[ v_{10}(t_1) + (t_2 - t_2^0) v_{11}(t_1) + \int_{t_2^0}^{t_2} \frac{(t_2 - \tau_2) v_{12}(t_1, \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} d\tau_2 \right] + e_{01}(t) \left[ v_{01} + \int_{t_1^0}^{t_1} \frac{v_{11}(\tau_1)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 + \right. \\ &+ \left. \int_{t_2^0}^{t_2} \frac{v_{02}(\tau_2)}{(\tau_2 - t_2^0)^{\alpha_1}} d\tau_2 + \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \frac{v_{12}(\tau_1, \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} d\tau_1 d\tau_2 \right] + e_{00}(t) \left[ v_{00} + (t_2 - t_2^0) v_{01} + \int_{t_1^0}^{t_1} \frac{1}{(\tau_1 - t_1^0)^{\alpha_1}} \times \right. \\ &\times \left. (v_{10}(\tau_1) + (t_2 - t_2^0) v_{11}(\tau_1)) d\tau_1 + \int_{t_2^0}^{t_2} \frac{(t_2 - \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} v_{02}(\tau_2) d\tau_2 + \right. \end{aligned} \tag{7}$$

$$\begin{aligned}
 & + \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \frac{(t_2 - \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 \Big] + e(t) \left[ v_{00} + (\bar{t}_2 - t_2^0) v_{01} + \int_{t_1^0}^{\bar{t}_1} \frac{1}{(\tau_1 - t_1^0)^{\alpha_1}} \times \right. \\
 & \quad \times (v_{10}(\tau_1) + (\bar{t}_2 - t_2^0) v_{11}(\tau_1)) d\tau_1 + \int_{t_2^0}^{\bar{t}_2} \frac{(\bar{t}_2 - \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} v_{02}(\tau_2) d\tau_2 + \\
 & \quad \left. + \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} \frac{(\bar{t}_2 - \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 = \psi_{12}(t), \quad t \in \Omega.. \right.
 \end{aligned}$$

Using the separation (3), the first boundary conditions in (2) can be written as

$$\begin{aligned}
 (l_{02}Uv)(t_2) & \equiv \beta_1 v_{02}(t_2) + \int_{t_1^0}^{t_1^1} \beta_2(t_1) v_{02}(t_2) dt_1 + \int_{t_1^0}^{t_1^1} \beta_2(t_1) \int_{t_1^0}^{t_1} \frac{v_{12}(\tau_1, t_2)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 dt_1 = \\
 & = \psi_{02}(t_2), \quad t_2 \in \Omega_2.
 \end{aligned}$$

Applying the Dirichlet formula, we obtain

$$\beta v_{02}(t_2) + \int_{t_1^0}^{t_1^1} \frac{\bar{\beta}_2(\tau_1) v_{12}(\tau_1, t_2)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 = \psi_{02}(t_2), \quad t_2 \in \Omega_2, \tag{8}$$

where  $\beta = \beta_1 + \bar{\beta}_2(t_1^0)$ ,  $\bar{\beta}_2(t_1) = \int_{t_1}^{t_1^1} \beta_2(s_1) ds_1$ .

If  $\beta \neq 0$ , then from equation (8) the function  $v_{02}(t_2)$  can be expressed as

$$v_{02}(t_2) = \frac{1}{\beta} \left[ \psi_{02}(t_2) - \int_{t_1^0}^{t_1^1} \frac{\bar{\beta}_2(\tau_1) v_{12}(\tau_1, t_2)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 \right], \quad t_2 \in \Omega_2. \tag{9}$$

By considering the equality (9) and the boundary conditions (2) into equation (7), we obtain

$$(l_{12}Uv)(t) \equiv v_{12}(t) + (C_1 v_{12})(t) + (C_2 v_{12})(t) = \Psi(t), \tag{10}$$

Where

$$\begin{aligned}
 (C_1 v_{12})(t) & \equiv \left[ \int_{t_1^0}^{t_1} \frac{e_{02}(t)}{(\tau_1 - t_1^0)^{\alpha_1}} v_{12}(\tau_1, t_2) d\tau_1 + \int_{t_2^0}^{t_2} \frac{e_{11}(t) + (t_2 - \tau_2) e_{10}(t)}{(\tau_2 - t_2^0)^{\alpha_2}} v_{12}(t_1, \tau_2) d\tau_2 + \right. \\
 & \quad \left. + \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \frac{e_{01}(t) + (t_2 - \tau_2) e_{00}(t)}{(\tau_1 - t_1^0)^{\alpha_1} \cdot (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 \right], \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 (C_2 v_{12})(t) &\equiv \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} \frac{(\bar{t}_2 - \tau_2) e(x)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 - \frac{1}{\beta} \times \\
 &\times \left[ \int_{t_1^0}^{\bar{t}_1} \frac{e_{02}(t) \bar{\beta}_2(\tau_1)}{(\tau_1 - t_1^0)^{\alpha_1}} v_{12}(\tau_1, t_2) d\tau_1 + \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} \frac{e_{01}(t) \bar{\beta}_2(\tau_1) + e_{00}(t) \bar{\beta}_2(\tau_1)(t_2 - \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} \times \right. \\
 &\quad \left. \times v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 + \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} \frac{e(t) \bar{\beta}_2(\tau_1)(\bar{t}_2 - \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 \right], \\
 \Psi(t) &\equiv \psi_{12}(t) - \frac{1}{\beta} e_{02}(t) \psi_{02}(t_2) - e_{11}(t) \psi_{11}(t_1) - e_{10}(t) [\psi_{10}(t_1) + (t_2 - t_2^0) \psi_{11}(t_1)] - \\
 &- e_{01}(t) \left[ \psi_{01} + \int_{t_1^0}^{\bar{t}_1} \frac{\psi_{11}(\tau_1)}{(\tau_1 - t_1^0)^{\alpha_1}} d\tau_1 + \frac{1}{\beta} \int_{t_2^0}^{\bar{t}_2} \frac{\psi_{02}(\tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} d\tau_2 \right] - e_{00}(t) [\psi_{00} + (t_2 - t_2^0) \psi_{01} + \\
 &+ \int_{t_1^0}^{\bar{t}_1} \frac{1}{(\tau_1 - t_1^0)^{\alpha_1}} (\psi_{10}(\tau_1) + (t_2 - t_2^0) \psi_{11}(\tau_1)) d\tau_1 + \frac{1}{\beta} \int_{t_2^0}^{\bar{t}_2} \frac{(t_2 - \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} \psi_{02}(\tau_2) d\tau_2 - \\
 &- e(t) \left[ \psi_{00} + (\bar{t}_2 - t_2^0) \psi_{01} + \int_{t_1^0}^{\bar{t}_1} \frac{1}{(\tau_1 - t_1^0)^{\alpha_1}} (\psi_{10}(\tau_1) + (\bar{t}_2 - t_2^0) \psi_{11}(\tau_1) + \right. \\
 &\quad \left. + \frac{1}{\beta} \int_{t_2^0}^{\bar{t}_2} \frac{(\bar{t}_2 - \tau_2)}{(\tau_2 - t_2^0)^{\alpha_2}} \psi_{02}(\tau_2) d\tau_2 \right)].
 \end{aligned}
 \tag{12}$$

Thus, we obtain the two-dimensional integral equation (10) for finding the function  $v_{12}$ . This leads to the following result.

**Theorem 1.** Suppose that  $\beta \neq 0$ . A necessary and sufficient condition for the operator  $l = (l_{12}, l_{02}, l_{11}, l_{10}, l_{01}, l_{00})$  to establish an isomorphism between the spaces  $W_{p,\alpha}^{(1,2)}(\Omega)$  and  $H_{p,\alpha}^{(1,2)}$  for an arbitrary function  $\Psi \in L_p(\Omega)$  is the uniqueness of the solution to the integral equation (10) corresponding to problem (1),(2).

Now, we investigate the solvability of the integral equation (10). Suppose that the function  $v_{12} \in L_p(\Omega)$  is a solution of the integral equation (10). Let us rewrite this equation in the following form of an operator equation

$$(I + C_1 + C_2)v_{12} = \Psi, \tag{13}$$

where  $I$  is the identity operator in the space  $L_p(\Omega)$ .

The operator  $C_1$  is a two-dimensional linear bounded Volterra integral operator acting in the space  $L_p(\Omega)$ . Therefore, the operator  $I + C_1$  has an inverse  $T = (I + C_1)^{-1}$  in the space  $L_p(\Omega)$ . Acting on both sides of equation (13) with the inverse operator  $T = (I + C_1)^{-1}$ , we obtain the solution

$$v_{12} + TC_2v_{12} = T\Psi. \tag{14}$$

Let us estimate the operator norm of  $\|C_2\|$ ,  $C_2 : L_p(\Omega) \rightarrow L_p(\Omega)$ . First, we estimate the norm of the first term of  $C_2$ .

$$\begin{aligned} & \left( \iint_{\Omega} \left| \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} \frac{e(t)(\bar{t}_2 - \tau_2)}{(\tau_1 - t_1^0)^{\alpha_1} (\tau_2 - t_2^0)^{\alpha_2}} v_{12}(\tau_1, \tau_2) d\tau_1 d\tau_2 \right|^p dt \right)^{\frac{1}{p}} \leq \left( \iint_{\Omega} |e(t)|^p dt \right)^{\frac{1}{p}} \left( \left( \int_{t_1^0}^{\bar{t}_1} (\tau_1 - t_1^0)^{-\alpha_1 q} d\tau_1 \right) \times \right. \\ & \left. \times \left( \int_{t_2^0}^{\bar{t}_2} (\tau_2 - t_2^0)^{-\alpha_2 q} (\bar{t}_2 - \tau_2)^q d\tau_2 \right) \right)^{\frac{1}{q}} \left( \int_{t_1^0}^{\bar{t}_1} \int_{t_2^0}^{\bar{t}_2} |v_{12}(\tau_1, \tau_2)|^p d\tau_1 d\tau_2 \right)^{\frac{1}{p}} \leq \frac{(\bar{t}_1 - t_1^0)^{1-\alpha_1}}{\sqrt[q]{1-q\alpha_1}} \times \\ & \times \frac{(\bar{t}_2 - t_2^0) \cdot (\bar{t}_2 - t_2^0)^{\frac{1}{q}-\alpha_2}}{\sqrt[q]{1-q\alpha_2}} \cdot \left( \iint_{\Omega} |e(t)|^p dt \right)^{\frac{1}{p}} \cdot \left( \iint_{\Omega} |v_{12}(t)|^p dt \right)^{\frac{1}{p}} = \\ & = \frac{(\bar{t}_1 - t_1^0)^{1-\alpha_1}}{\sqrt[q]{1-q\alpha_1}} \frac{(\bar{t}_2 - t_2^0)^{\frac{1}{q}-\alpha_2+1}}{\sqrt[q]{1-q\alpha_2}} \|e\|_{L_p(\Omega)} \cdot \|v_{12}\|_{L_p(\Omega)}. \end{aligned}$$

Similarly, estimating the remaining terms of the operator  $C_2$ , we obtain the following inequality

$$\|C_2v_{12}\|_{L_p(\Omega)} \leq k \cdot \|v_{12}\|_{L_p(\Omega)}, \tag{15}$$

Where

$$k = \frac{\|\bar{\beta}_2(\cdot)\|_{C(\bar{\Omega}_1)}}{|\beta|} \frac{(\bar{t}_1^1 - t_1^0)^{1-\alpha_1}}{\sqrt[q]{1-q\alpha_1}} \frac{(\bar{t}_2^1 - t_2^0)^{1-\alpha_2}}{\sqrt[q]{1-q\alpha_2}} \left( \|e_{01}\|_{L_p(\Omega)} + (\bar{t}_2^1 - t_2^0) \left( \|e_0\|_{L_p(\Omega)} + \|e\|_{L_p(\Omega)} \right) \right) +$$

$$+ \frac{(t_1^1 - t_1^0)^{\frac{1}{q} - \alpha_1}}{\sqrt[q]{1 - q\alpha_1}} \left( \frac{(t_2^1 - t_2^0)^{\frac{1}{q} - \alpha_2 + 1}}{\sqrt[q]{1 - q\alpha_2}} \|e\|_{L_p(\Omega)} + \frac{\|\bar{\beta}_2(\cdot)\|_{C(\bar{\Omega}_1)}}{|\beta|} \cdot \|e_{02}^0(\cdot)\|_{L_p(\Omega_1)} \right)$$

Hence,

$$\|C_2\| \leq k .$$

Then

$$\|TC_2 v_{12}\|_{L_p(\Omega)} \leq k \|T\| \|v_{12}\|_{L_p(\Omega)} .$$

If

$$k_1 = \|T\| k < 1,$$

then the integral equation (10) has a unique solution  $v_{12} \in L_p(\Omega)$  for any function  $\Psi \in L_p(\Omega)$ . Moreover, the solution of equation (10) satisfies the following inequality

$$\|v_{12}\|_{L_p(\Omega)} \leq \frac{1}{1 - k_1} \|T\| \|\Psi\|_{L_p(\Omega)} . \tag{16}$$

**Theorem 2.** If  $\beta \neq 0$  and  $k_1 < 1$  are given, then equation (13) has a unique solution  $v_{12} \in L_p(\Omega)$  for any  $\Psi \in L_p(\Omega)$ , and the estimate (16) is valid for this solution.

Based on Theorems 1 and 2, we obtain the following result.

**Theorem 3.** If  $\beta \neq 0$  and  $k_1 < 1$  are given, then problem (1),(2) is well-posed.

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