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BASIS PROPERTY OF ROOT FUNCTIONS OF SOME EIGENVALUE PROBLEM WITH SPECTRAL PARAMETER CONTAINED IN THE BOUNDARY CONDITION

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Abstract

In this paper, we consider the boundary value problem which describes the small bending vibrations of homogeneous beam in cross-sections of which the longitudinal force acts, the left end of which is fixed, and the mass is concentrated on the right end. The oscillatory properties of eigenfunctions are studied and the basis property in $L_p(0, 1),\ 1 < p < \infty$, of the system of root functions without one arbitrary remote function is established.

Keywords: boundary value problem, bending vibrations, eigenvalue, eigenfunction, basis property

Mathematics Subject Classification *(2020): 34B05, 34B08, 34C10, 34C23, 34L10, 47A75, 74H45*

1. Introduction

We consider the following spectral problem

$$\ell(y) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), x \in (0, 1), \tag{1}$$

$$y(0) = y'(0) = y''(1) = 0, \tag{2}$$

$$Ty(1) = a\lambda y(1), \tag{3}$$

where $\lambda \in C$ is an eigenvalue parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on $[0, 1]$ and a is some positive constant.

Problem (1)-(3) arises when separating variables in a boundary value problem describing small bending vibrations of a homogeneous rod, the left end of which is rigidly fixed, and at the right end there is a particle of mass c. In particular, the case $a < 0$ corresponds to the situation when a tracking force acts on the right end of the rod (see, for example, [6] and [9]) .

Note that in case of $a < 0$, problem (1)-(3), when the boundary condition (3) has a more general form, was investigated in the paper [7] (see also [8]), where it was shown that the eigenvalues of this problem are real, simple and form an infinitely increasing sequence. The oscillatory properties of eigenfunctions are also studied, with the help of which asymptotic formulas for eigenvalues and eigenfunctions are obtained. In addition, it is proved that the system of eigenfunctions without any arbitrary remote function forms a basis in the space $L_p(0,1),\ 1 < p < \infty.$

In the case of $a > 0$ problem (1)-(3), when the boundary conditions has a more general form, was investigated in the paper [1]. In this case either all eigenvalues are real and simple, or all eigenvalues are simple and all except a conjugate pair of non-real are real, or all eigenvalues are real and all except one double or one triple are simple. In this paper we show that all eigenvalues of problem (1)-(3) are real, simple and forman unboundedly increasing sequence. Moreover, we study oscillation properties and basis properties in $L_p(0,1),\ 1 < p < \infty,$ of eigenfunctions of this problem.

2. Preliminary

Lemma 1 [7, Theorem 2.1]. *For each fixed $\lambda \in C$ there exists a nontrivial solution $y(x,\lambda)$ of problem (1), (2) up a constant factor.*

Remark 1. We can choose the solution $y(x,\lambda)$ of problem (1), (2) so that it is an entire function of parameter λ for each $x \in [0,1]$.

Alongside the spectral problem (1)-(3) we consider the spectral problem

$$\begin{cases} y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), x \in (0,\ 1), \\ y(0) = y'(0) = y''(1) = 0, \\ y(1)\cos\delta - Ty(1)\sin\delta = 0, \end{cases} \tag{4}$$

where $\delta \in [0, \pi/2]$. It follows from [5, Theorems 5.4 and 5.5] that the eigenvalues of problem (4) are real, simple and form an infinitely increasing sequence $\{\lambda_k(\delta)\}_{k=1}^{\infty}$; for each $k \in \mathrm{N}$ the eigenfunction $y_{k,\delta}(x)$ corresponding to the eigenvalue $\lambda_k(\delta)$ has exactly $k-1$ simple zeros in the interval $(0, 1)$.

It follows from [5, Property 1] that the following relation holds:

$$0 < \lambda_1(\pi/2) < \lambda_1(0) < \lambda_2(\pi/2) < \lambda_2(0) < \ldots < \lambda_k(\pi/2) < \lambda_k(0) < \ldots \quad (5)$$

Obviously, the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2)$ are zeros of entire functions $y(1,\lambda)$ and $Ty(1,\lambda)$ respectively. It follows from (5) that the zeros of functions $y(1,\lambda)$ and $Ty(1,\lambda)$ do not coincide. Then the function $F(\lambda) = \frac{Ty(1,\lambda)}{y(1,\lambda)}$ is defined in $D = (C \setminus R) \cup \bigcup_{k=1}^{\infty} (\lambda_{k-1}(0), \lambda_k(0))$, where $\lambda_0(0) = -\infty$, and is a meromorphic function of finite order.

By [7, Lemmas 3.1 and 3.2] we have the following relations:

$$\frac{dF}{d\lambda} = \frac{1}{y^2(1,\lambda)} \int_0^1 y^2(x,\lambda)\, dx, \quad (6)$$

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty. \quad (7)$$

Let $m(\lambda)$ denote the number of zeros of the function $y(x,\lambda)$ contained in the interval $(0, 1)$. Then it follows from [2, Lemma 4 and formula (24)] (see also [7, Theorem 3.1]) and [8, Lemma 2.3]) that

$$m(\lambda) = k-1 \text{ for } \lambda \in [0, +\infty) \cap (\lambda_{k-1}(0), \lambda_k(0)], \quad k \in \mathrm{N}, \quad (8_1)$$

$$m(\lambda) = \sum_{\varsigma_k \in (\lambda, 0)} i(\varsigma_k) \text{ for } \lambda \in (-\infty, +\infty), \quad (8_2)$$

where $i(\varsigma_k)$ is the oscillation index (see [3, p. 2323]) of the eigenvalue $\varsigma_k, k \in \mathrm{N}$, of the spectral problem

$$\begin{cases} y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), x \in (0, 1), \\ y(0) = y'(0) = y''(0) = y''(1) = 0. \end{cases} \quad (9)$$

It follows from [3, Theorem 4.1] that there exists $\varsigma < 0$ such that the eigenvalues $\varsigma_k, k \in \mathrm{N}$, of problem (9), which lie on the interval $(-\infty, \varsigma)$ and numbered in decreasing order, are simple and have oscillation index 1.

By (5) and (6) we get $F(0) < 0$. Moreover, in view of [1, Lemma 3.1] we have

$$F(\lambda) = F(0) + \sum_{k=1}^{\infty} \frac{\lambda c_k}{\lambda_k(0)(\lambda - \lambda_k(0))}, \tag{10}$$

where $c_k = \underset{\lambda=\lambda_k(0)}{res} F(\lambda) < 0, k \in \mathrm{N}.$ Then it follows from (10) that

$$F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3},$$

whence implies that

$$F''(\lambda) > 0 \quad \text{for} \quad \lambda \in (-\infty, \lambda_1(0)), \tag{11}$$

i.e. the function $F(\lambda)$ is convex in the interval $(-\infty, \lambda_1(0)).$

3. Main properties of eigenvalues and eigenfunctions of problem (1)-(3)

Lemma 1. *The eigenvalues of problem (1)-(3) are real, simple and form an at most countable set without finite limit points.*

Proof. It is obvious that the eigenvalues of problem (1)-(3) are the roots of the equation

$$Ty(1,\lambda) = a\lambda y(1,\lambda). \tag{12}$$

If $\lambda \in C \setminus R$ is an eigenvalue of problem (1)-(3), then $\bar{\lambda}$ is also eigenvalue of this problem since the coefficients $q(x)$ and a are real. Note that $y(x,\bar{\lambda}) = \overline{y(x,\lambda)},$ and consequently, equality (11) holds for $\bar{\lambda},$ if it holds for $\lambda.$

By (12) it follows from [7, formula (4.2)] that

$$a(\bar{\lambda} - \lambda)\,|\,y(1,\lambda)\,|^2 = (\bar{\lambda} - \lambda)\int_0^1 |\,y(x,\lambda)\,|^2 dx.$$

Since $\lambda \in C \setminus R$ from the last relation we get

$$\int_0^1 |\,y(x,\lambda)\,|^2 dx - a\,|\,y(1,\lambda)\,|^2 = 0. \tag{13}$$

On the other hand, multiplying both parts of (1) by $\overline{y(x,\lambda)},$ integrating the resulting relation in the range from 0 to $1,$ applying the formula for integration by parts and taking into account the boundary conditions (2) and (3) we obtain

$$\int_0^1 \{|\,y''(x,\lambda)\,|^2 + q(x)\,|\,y'(x,\lambda)\,|^2\}\,dx = \lambda\left\{\int_0^1 |\,y(x,\lambda)\,|^2 dx - a\,|\,y(1,\lambda)\,|^2\right\}, \tag{14}$$

whence, by (13), we get

$$\int_0^1 \{|\,y''(x,\lambda)\,|^2 + q(x)\,|\,y'(x,\lambda)\,|^2\}\,dx = 0.$$

It follows from last relation that $y(x,\lambda)\equiv const,$ and consequently, by (2) we have $y(x,\lambda)\equiv 0,$ which contradicts the fact that λ is an eigenvalue of problem (1)-(3).

Recall that the zeros of the entire function on the left-hand side of Eq. (12) is eigenvalues of the boundary-value problem (1)-(3) which are real. Consequently, this function does not vanish for nonreal λ. Hence it does not vanish identically and therefore, its zeros form an at most countable set without finite limit points.

Remark 1. Note that if λ is an eigenvalue of problem (1)-(3) and $y(1,\lambda)=0,$ then $Ty(1,\lambda)=0$ which contradicts the relation (5).

By Remark 1 the equation (11) is equivalent to the equation

$$F(\lambda)=a\lambda. \tag{15}$$

If λ is a multiple eigenvalue of problem (1)-(3), then we have

$$F(\lambda)=a\lambda \quad \text{and} \quad F'(\lambda)=a. \tag{16}$$

Hence it follows from (6) that

$$\int_0^1 y^2(x,\lambda)\,dx - a\,y^2(1,\lambda)=0. \tag{17}$$

Since $\lambda\in R$ by (17) from (14) we obtain

$$\int_0^1 \{y''^2(x,\lambda)+q(x)y'^2(x,\lambda)\}\,dx=0,$$

whence implies that $y(x,\lambda)\equiv 0,$ a contradiction. The proof of this lemma is complete.

Lemma 2. *Eq. (15) can have only one root in the interval* $(\lambda_{k-1}(0),\lambda_k(0))$ *for* $k\in\mathrm{N},\ k\geq 2.$

Proof. Let Eq. (15) have at least two roots λ_{k1}^* and λ_{k2}^*, $\lambda_{k1}^*<\lambda_{k2}^*$ (we can assume that these zeros are consecutive and λ_{k1}^* is the closest to $\lambda_{k-1}(0)$) in the interval $(\lambda_{k-1}(0),\lambda_k(0))$ for some $k\geq 2$. By (6), the function $F(\lambda)$ is continuous in the interval $(\lambda_{k-1}(0),\lambda_k(0))$ and, increasing, takes values from $-\infty$ to $+\infty$. Since $a>0$ the function $G(\lambda)=a\lambda$ also increases in this interval. Hence

$$F(\lambda)-a\lambda<0 \text{ for } \lambda<\lambda_{k1}^*,\ F(\lambda)-a\lambda>0 \text{ for } \lambda_{k1}^*<\lambda<\lambda_{k2}^* \text{ and}$$
$$F(\lambda)-a\lambda<0 \text{ for } \lambda>\lambda_{k2}^*,$$

whence implies that

$$F'(\lambda_{k1}^*)-a>0 \text{ and } F'(\lambda_{k2}^*)-a<0.$$

On the other hand, since $\lambda_{k2}^* > 0$ from (14) we obtain

$$\int_0^1 y^2(x, \lambda_{k2}^*)\, dx - a\, y^2(1, \lambda_{k2}^*) > 0. \tag{18}$$

In view of (18) by (6) we get

$$F'(\lambda_{k2}^*) - a > 0,$$

a contradiction. The proof of this lemma is complete.

Theorem 1. *The eigenvalues of problem (1)-(3) form an unboundedly increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that*

$$\lambda_1 \in (-\infty, 0) \ \textit{ and } \ \lambda_k \in (\lambda_{k-1}(\pi/2), \lambda_k(0)) \ \textit{ for } \ k \geq 2. \tag{19}$$

Moreover, the eigenfunctions $y_k(x),\ k \in \mathrm{N}$, *corresponding to the eigenvalues* $\lambda_k, k \in \mathrm{N}$, *have the following oscillation properties: the function* $y_k(x)$ *for* $k \geq 2$ *has exactly* $k-1$ *simple zeros, the function* $y_1(x)$ *has* $\sum\limits_{\varsigma_k \in (\lambda_1, 0)} i(\varsigma_k)$ *simple zeros in the interval* $(0, 1)$.

Proof. By (6), (7) and (10) we have

$$\lim_{\lambda \to \lambda_k(0)-0} F(\lambda) = +\infty, \ \lim_{\lambda \to \lambda_{k-1}(0)+0} F(\lambda) = -\infty, \ F(\lambda_k(\pi/2)) = 0, \tag{20}$$

$F(\lambda) < 0$ for $(\lambda_{k-1}(0), \lambda_k(\pi/2))$ and $F(\lambda) > 0$ for $(\lambda_k(\pi/2), \lambda_k(0))$, $k \in \mathrm{N}$. (21)

By (11) the function $F(\lambda)$ is convex in the interval $(-\infty, \lambda_1(0))$. Since the function $G(\lambda) = a\lambda$ is increasing in this interval, by (20) and (21) it follows that Eq. (15) has two solution $\lambda_1 \in (-\infty, 0)$ and $\lambda_2 \in (\lambda_1(\pi/2), \lambda_1(0))$. Therefore, according to (8_2) and (8_1), the number of zeros of the function $y_1(x)$ contained in $(0, 1)$ is equal to $\sum\limits_{\varsigma_k \in (\lambda_1, 0)} i(\varsigma_k)$ and the function $y_2(x)$ has no zeros in $(0, 1)$.

By the first two relations of (20) and Lemma 2 Eq. (15) has unique root λ_{k+1} in $(\lambda_{k-1}(0), \lambda_k(0))$ for each $k \geq 2$. Moreover, by the second relation of (21) we have $\lambda_{k+1} \in (\lambda_k(\pi/2), \lambda_k(0))$. Then by ($8_1$) the eigenfunction $y_{k+1}(x)$ for $k \geq 2$ has exactly $k-1$ simple zeros in the interval $(0, 1)$. The proof of this theorem is complete.

In view of [1, Theorem 5.1] (see also [7, Theorem 6.1]) we have the following asymptotic formulas for eigenvalues and eigenfunctions of problem (1)-(3).

Theorem 2. *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{3}{4}\right)\pi + O\left(\frac{1}{k}\right), \tag{22}$$

$$y_k(x) = \sin\left(k - \frac{3}{4}\right)\pi x - \cos\left(k - \frac{3}{4}\right)\pi x + e^{-\left(k-\frac{3}{4}\right)\pi x} + O\left(\frac{1}{k}\right), \qquad (23)$$

where relation (23) holds uniformly for $x \in [0, 1]$.

4. Basis properties of eigenfunctions of problem (1)-(3)

Let $H = L_2(0,1) \oplus C$ be the Hilbert space with scalar product

$$(\hat{y}, \hat{\vartheta})_H = (\{y, m\}, \{\vartheta, n\})_H = \int_0^1 y(x)\,\overline{\vartheta(x)}\,dx + a^{-1} m\bar{n}. \qquad (24)$$

It is obvious that problem (1)-(3) reduces to the spectral problem for the operator $L: D(L) \subset H \to H$ which is defined as follows:

$$L\hat{y} = L\{y, m\} = \{\ell(y), ay(1)\},$$

$$D(L) = \{\hat{y} = \{y, m\} \in H : y \in W_2^4(0,1), \ell(y) \in L_2(0, 1), y(0) = y'(0) = y''(1) = 0, m = ay(1)\}.$$

Note that the operator L is well defined, and consequently, problem (1)-(3) can be rewritten in the following equivalent form

$$L\hat{y} = \lambda\hat{y}, \hat{y} \in D(L). \qquad (25)$$

In this case the eigenvalues $\lambda_k, k \in \mathrm{N}$, of problem (1)-(3) and those of problem (25) coincide together with their multiplicities; moreover, there exists a one-to-one correspondence between the eigenfunctions of the two problems,

$$\hat{y}_k = \{y_k, ay_k(1)\} \leftrightarrow y_k,\ k \in \mathrm{N}.$$

Let $J: H \to H$ be the operator defined by

$$J\hat{y} = J\{y, m\} = \{y, -m\}.$$

Then this oparator generates the Pontryagin space $\Pi_1 = L_2(0,1) \oplus C$ with inner product (see [4])

$$(\hat{y}, \hat{\vartheta})_{\Pi_1} = (\{y, m\}, \{\vartheta, n\})_H = \int_0^1 y(x)\,\overline{\vartheta(x)}\,dx - a^{-1} m\bar{n}.$$

Theorem 3 [1, Theorem 2.1 and Lemma 2.1]. *The operator* L *is* J *-self-adjoint in* Π_1. *If* L^* *is the adjoint operator of* L *in* H, *then* $L^* = JLJ$.

In view of Lemma 1 we have

$$L\hat{y}_k = \lambda_k y_k \text{ for any } k \in \mathrm{N}. \qquad (26)$$

By $\{\hat{\vartheta}_k^*\}_{k=1}^{\infty}, \hat{\vartheta}_k^* = \{\vartheta_k^*, s_k^*\}$, we denote the system of eigenfunctions of the operator L^*, i.e.,

$$L^* \hat{\vartheta}_k^* = \lambda_k \hat{\vartheta}_k^* \text{ for any } k \in \mathrm{N}. \tag{27}$$

Then by Theorem 3 it follows from (26) and (27) that

$$\vartheta_k^* = J\hat{y}_k \text{ for any } k \in \mathrm{N}. \tag{28}$$

Let

$$\tau_k = (\hat{y}_k, \hat{y}_k)_{\Pi_1} = \int_0^1 y_k^2(x)dx - a^{-1}m_k^2 = \int_0^1 y_k^2(x)dx - ay_k^2(1). \tag{29}$$

It follows from Remark 1 and [1, Lemma 6.1] that

$$m_k \neq 0 \text{ and } \tau_k \neq 0 \text{ for any } k \in \mathrm{N}. \tag{30}$$

Hence by [1, Lemma 6.3] and (28) the elements of the system $\{\hat{\vartheta}_k\}_{k=1}^\infty, \hat{\vartheta}_k = \{\vartheta_k, s_k\}$, adjoint to the system $\{\hat{y}_k\}_{k=1}^\infty$ are defined as follows:

$$\hat{\vartheta}_k = \tau_k^{-1}\hat{\vartheta}_k^* = \tau_k^{-1}Jy_k = \tau_k^{-1}\{y_k, -m_k\},$$

and consequently, by (30), we have

$$s_k = -\tau_k^{-1}m_k \neq 0 \text{ for any } k \in \mathrm{N}.$$

Then follows from [1, Theorem 6.2] the following result.

Theorem 4. *Let* r *be an arbitrary natural number. Then the system* $\{y_k\}_{k=1, k\neq r}^\infty$ *of eigenfunctions of problem (1)-(3) forms a basis in the space* $L_p(0, 1)$, $1 < p < \infty$, *which is an unconditional basis for* $p = 2$. *Moreover, the system* $\{u_k\}_{k=1, k\neq r}^\infty$ *adjoint to the system* $\{y_k\}_{k=1, k\neq r}^\infty$ *is determined as follows:*

$$u_k(x) = \vartheta_k(x) - \frac{s_k}{s_r}\vartheta_r(x) = \tau_k^{-1}\left\{y_k - \frac{m_k}{m_r}y_r\right\}.$$

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