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ON RIEMANN'S METHOD FOR THE DISCRETE ANALOGUE OF A SECOND-ORDER HYPERBOLIC EQUATION

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Abstract

In this paper is considered the Cauchy problem for a discrete analogue of a second-order hyperbolic equation with a periodic coefficient. Using the eigenfunctions of the discrete Hill equation, the Riemann function of a discrete analogue of a second-order hyperbolic equation is constructed. A representation of the solution to the Cauchy problem through Riemann functions is found.

Keywords: hyperbolic equation is, Riemann function, Riemann method. Cauchy problem, discrete Hill equation.

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1. Introduction

The Riemann method of representing solutions to various boundary value problems is

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well known in the theory of linear differential equations of hyperbolic type with two independent variables [1], [3]-[8]. The essence of this method is that the value of the desired function at a given point may be thought of as the value of some linear functional on the initial data. An expression for this functional was first found by Riemann. The Riemann function plays a fundamental role in the theory of linear equations of hyperbolic type and with its help it is usually possible to write down the solution of Goursat problems in quadratures. Let $p(x)$ be a continuous real function such that $p(x+1)=p(x)$, and let $f(x)$ be a twice continuously differentiable compactly supported function. Under these assumptions about $p(x)$ and $f(x)$, consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - p(t)u = \frac{\partial^2 u}{\partial x^2} - p(x)u\,, \qquad (1)$$

$$u\big|_{t=t_0} = 0,\ \frac{\partial u}{\partial t}\bigg|_{t=t_0} = f(x), \qquad (2)$$

where t_0 is any fixed number. It is known that one of the main tools for studying the Cauchy problem

for a second-order hyperbolic equation is the application of the Riemann function method. To apply

this method, we must construct the Riemann function $R(x,t,x_0,y_0)$, that is, a twice continuously differentiable solution of the equation

$$\frac{\partial^2 R}{\partial t^2} - p(t)R = \frac{\partial^2 R}{\partial x^2} - p(x)R\,,$$

taking the value 1 on the characteristics $x - x_0 = \pm(t - t_0)$ of this equation(see [5]). It is well known [5], [6] that, using the Riemann function, one can represent the solution of problem (1), (2) as

$$U(X,T) = \pm\frac{1}{2}\int_{X-T+t_0}^{X+T-t_0} f(x)R(x,t_0,X,T)dx\,, \qquad (3)$$

the signs $\pm$ correspond to the cases $\pm(T - t_0) > 0$. On the other hand, in papers [3], [4] the inverse spectral problem for the perturbed discrete Hill equation was investigated using the transformation operator method. As follows from [2] the kernel of the transformation operator satisfies an equation of the following form

$$U(n+1,m)+(b_n+c_n)U(n,m)+U(n-1,m)=$$
$$=U(n,m+1)+(b_m+c_m)U(n,m)+(n,m-1), m>n+1,$$

$$U(n,n+1)-U(n-1,n)=c_n,$$

$$U(n,n+2)-U(n-1,n+1)+(b_{n+1}-b_n-c_n)U(n,n+1)=0.$$

Consider the following problem

$$\begin{aligned}&U(n+1,m)+b_n U(n,m)+U(n-1,m)=\\&=U(n,m+1)+b_m U(n,m)+U(n,m-1),\\&n=0,\pm1,\pm2,\ldots, m>k_0\end{aligned} \tag{4}$$

$$U(n,k_0)=0,\ U(n,k_0+1)=f(n) \tag{5}$$

where b_n is real coefficient satisfying the condition $b_{n+N}=b_n$ with N a natural number. We will call the function $R(n,m,n_0,m_0)$ the Riemann function of equation (4) if it satisfies equation (4) and taking the value 1 on the characteristics $n-n_0=\pm(m-m_0+1)$ of this equation.

In the present paper the Riemann function of equation (1)) is investigated. An explicit form of the Riemann function is found in terms of Floquet solutions of the discrete Hill equation.

2. Construction of the Riemann function

We need some preliminaries concerning the discrete Hill equation. Consider the equation

$$y_{n-1}+b_n y_n+y_{n+1}=\lambda y_n, n=0,\pm1,\pm2,\ldots, \tag{6}$$

where λ is a complex parameter. For $N=1$ this problem was considered in

[2]; therefore, we suppose henceforth that $N > 1$. Let

$$\chi(1,0;\lambda) = \chi(N, N-1;\lambda) \equiv 1,$$

$$\chi(m,s;\lambda) = (-1)^{s+m+1} \begin{vmatrix} b_m - \lambda & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & b_{m+1} - \lambda & 1 \dots & 0 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & b_{s-1} - \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & b_s - \lambda \end{vmatrix},\ .$$

$1 \le m \le s \le N-1$

$$\Delta_{\pm}(\lambda) = \begin{vmatrix} b_0 - \lambda & 1 & 0 & \dots & 0 & 0 & \pm 1 \\ 1 & b_1 - \lambda & 1 \dots & 0 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 1 & b_{N-2} - \lambda & 1 \\ \pm 1 & 0 & 0 & \dots & 0 & 1 & b_{N-1} - \lambda \end{vmatrix},$$

$$\mu = \Delta_{\pm}(\lambda) \pm (-1)^N 2 .$$

Denote the roots of the polynomials $\Delta_+(\lambda)$ and $\Delta_-(\lambda)$– by $\lambda_1^* < \lambda_2^* \le \dots \le \lambda_N^*$ and $\lambda_1^- < \lambda_2^- \le \dots \le \lambda_N^-$. As is well known [2], these roots alternate, and we can arrange them as follows:

$$\lambda_1^* < \lambda_1^- \le .\lambda_2^-. < \lambda_2^*. \le \lambda_3^* < \dots < \lambda_{N-1}^* \le \lambda_N^- < \lambda_N^* \text{ , if } N \text{ is even,}$$

$$\lambda_1^- < \lambda_1^+ \le .\lambda_2^+. < \lambda_2^-. \le \lambda_3^* < \dots < \lambda_{N-1}^* \le \lambda_N^- < \lambda_N^* \text{ , if } N \text{ is odd.}$$

Let Γ be the complex λ-plane with cuts along the segments $I_1, I_2, \dots, I_N$, where I_j has endpoints $\lambda_j^{\pm}$, $j = 1,2,\dots,N$. We introduce the function

$$z = z(\lambda) = \frac{(-1)^N \mu}{2} + \sqrt{\frac{\mu^2}{4} - 1},$$

taking a regular branch of $z(\lambda)$ in Γ such that $z(\infty)=0$. Using the definition of z(_), we can prove as in [2] that

$$z(\lambda)=\frac{A}{\lambda^N}+O\left(\frac{1}{\lambda^{N+1}}\right),\ \lambda\to\infty.$$

As shown in [5], equation (6) has solutions $\psi_n^{\pm}(\lambda)$, which can be represented as

$$\psi_{nN+j-1}^{\pm}(\lambda)=E_j^{\pm}(\lambda)z^{\pm n},\quad j=1,\ldots,N,\ |n|=0,1,2,\ldots, \tag{7}$$

where

$$E_1^{\pm}(\lambda)\equiv 1,\quad E_j^{\pm}(\lambda)=\chi^{-1}(1,N-1;\lambda)\{\chi(1,j-2;\lambda)z^{\pm 1}+\chi(j,N-1;\lambda)\},\ j=2,..,N.$$

Let us introduce the notations

$$\rho(\lambda)=\frac{\chi(1,N-1;\lambda)}{z-z^{-1}}.$$

$$F(n,k,m,r)=\frac{1}{2\pi i}\int_{\partial\Gamma}\rho^2(\lambda)\left[\psi_n^+(\lambda)\psi_k^-(\lambda)-\psi_k^+(\lambda)\psi_n^-(\lambda)\right]\psi_m^+(\lambda)\psi_r^-(\lambda)d\lambda,k>n\ \cdot \tag{8}$$

Theorem 1. *The following equalities hold:*

$$F(n,k,m,r)=0,\ \pm r\geq\pm(m\pm(k-n)),$$

$$F(n,k,m,r)=-1,\ \pm r=\pm(m\pm(k-n)\mp 1),$$

Proof. It follows from the definition of $F(n,k,m,r)$ and the residue theorem that

$$F(n,k,m,r)=\operatorname*{res}_{\lambda=\infty}\rho^2(\lambda)\left[\psi_n^+(\lambda)\psi_k^-(\lambda)-\psi_k^+(\lambda)\psi_n^-(\lambda)\right]\psi_m^+(\lambda)\psi_r^-(\lambda)=$$

$$=\operatorname*{res}_{\lambda=\infty}\rho^2(\lambda)\psi_n^+(\lambda)\psi_k^-(\lambda)\psi_m^+(\lambda)\psi_r^-(\lambda)-\operatorname*{res}_{\lambda=\infty}\rho^2(\lambda)\psi_k^+(\lambda)\psi_n^-(\lambda)\psi_m^+(\lambda)\psi_r^-(\lambda).$$

Let $r\leq m-(k-n)$ and $k>n$. Suppose that $n=n_1N+n_2-1,\ k=k_1N+k_2-1,$ $m=m_1N+m_2-1,\ \ r=r_1N+r_2-1,$ where the numbers n_2,k_2,m_2 and r_2 vary from 1 to N. By (4), we find up to a constant factor that

$$\psi_n^+(\lambda)\psi_k^-(\lambda)\psi_m^+(\lambda)\psi_r^-(\lambda)=\lambda^{k-n+r-m}(1+o(1)),\ \lambda\to\infty.$$

From the definition of $\rho(\lambda)$ we now obtain

$$\rho(\lambda) \sim \frac{1}{\lambda},\ \lambda \to \infty .$$

Recalling that $k-n+r-m \le 0$, from the last two relations we conclude that

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi_n^+(\lambda)\psi_k^-(\lambda)\psi_m^+(\lambda)\psi_r^-(\lambda) = 0 .$$

On other hand, if $k-n+r-m=1$, then

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi_n^+(\lambda)\psi_k^-(\lambda)\psi_m^+(\lambda)\psi_r^-(\lambda) = -1 .$$

Since the condition $k-n+r-m \le 1$, implies $n-k+r-m \le 0$, we similarly find that

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi_k^+(\lambda)\psi_n^-(\lambda)\psi_m^+(\lambda)\psi_r^- = 0 .$$

Thus,

$$F(n,k,m,r) = 0$$

for $r \le m-(k-n)$ and

$$F(n,k,m,r) = -1$$

for $r = m-(k-n)+1$. Moreover, since the function $\rho^2(\lambda)$ takes real values for $\lambda \in \partial\Gamma$ and $\psi_k^+(\lambda)$ while $\psi_k^-(\lambda)$ are complex conjugates, passing to the conjugate expressions in the last formula, we arrive at

$$F(n,k,r,m) = 0$$

for $m \ge r+(k-n)$ and

$$F(n,k,r,m) = -1$$

for $m = r+(k-n)-1$.

The theorem is proved.

Now we introduce the function

$$R(n,m,n_0,m_0) = -F(m,m_0,n_0,n) . \tag{9}$$

Theorem 2. *The solution of the problem (4), (5) can be represented as*

$$U(n,m)=\sum_{j=n-(m-k_0)+1}^{n+(m-k_0)-1} f(j)R(j,k_0,n,m) \ , \tag{10}$$

where the sum value is considered to be zero if the upper limit of the summation is less than the lower limit.

Proof. As noted above, the functions $\psi_n^+(\lambda),\psi_n^-(\lambda)$ are solutions of equation (6). From this and from (8), (9) it follows that $R(n,m,n_0,m_0)$ satisfies equation (4). By virtue of Theorem 1, the function $R(n,m,n_0,m_0)$ taking the value 1 on the characteristics $n-n_0=\pm(m-m_0+1)$. In other words, $R(n,m,n_0,m_0)$ is the Riemann function of equation (4). Then it is directly verified that function (10) is a solution to problem (4), (5). The theorem is proved.

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