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UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS OF SOME EIGENVALUE PROBLEM WITH SPECTRAL PARAMETER CONTAINED IN THE BOUNDARY CONDITION

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Abstract

This paper considers an eigenvalue problem for ordinary differential equations of fourth order with a spectral parameter in one of the boundary conditions. Sufficient conditions are established for the uniform convergence of Fourier series expansions in the system of eigenfunctions of this problem.
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1. Introduction

We consider the following eigenvalue problem

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$$y^{(4)}(x)-(q(x)y'(x))'=\lambda y(x), x\in(0,1), \tag{1}$$

$$y''(0)=y(1)=y'(1)==0, \tag{2}$$

$$Ty(0)=a\lambda y(0), \tag{3}$$

where $\lambda\in C$ is an eigenvalue parameter, $Ty\equiv y'''-qy'$, q is a positive absolutely continuous function on $[0,1]$ and a is a nonzero constant.

In the case $a<0$ problem (1)-(3) arises when separating variables in a boundary value problem describing small bending vibrations of a homogeneous rod, in the cross sections of which a longitudinal force acts, the right end of which is rigidly fixed, and at the left end there is a particle of mass a (see [4, 5, 10]).

Eigenvalue problems for ordinary differential equations of fourth order with a spectral parameter in the boundary conditions have been considered in many papers (see, for example, [1-3, 6-8] and references therein) in various formulations. In these papers, the general arrangement of eigenvalues on the real axis, the structure of root subspaces, the oscillatory properties of eigenfunctions were studied, and asymptotic formulas for eigenvalues and eigenfunctions were obtained. Moreover, using these properties, the basis properties of root functions in the space $L_p, 1<p<\infty,$ was investigated. In [1] and [3] sufficient conditions are established for the uniform convergence of Fourier series of continuous functions with respect to the system of root functions of the problems under consideration.

In this paper, we find sufficient conditions for the uniform convergence of Fourier series expansions of a continuous function in the system of eigenfunctions of problem (1)-(3).

2. Preliminary

By changing the variables $t=1-x$, we transform problem (1)-(3) into the following spectral problem

$$\begin{gathered}\vartheta^{(4)}(x)-(\breve{q}(t)\,\vartheta'(t))'=\lambda\vartheta(t), t\in(0,1),\\ \vartheta(0)=\vartheta'(0)=0, \vartheta''(1)=0\\ \breve{T}\vartheta(1)=-a\lambda\vartheta(1),\end{gathered} \tag{4}$$

where $\vartheta(t) = y(x(t)) = y(1-t)$, $\breve{q}(t) = q(x(t)) = q(1-t)$, $\breve{T}\vartheta(t) = T\vartheta(x(t)) = \vartheta'''(t) - -\breve{q}(t)\,\vartheta(t)$, $t \in [0,1]$. Note that problem (4) in the case $a > 0$ in a general form is considered in [7, 8], and in the case $a < 0$ is considered in [2] (see also [6]).

Theorem 1 ([8, Theorem 2.2] and [6, Theorem 1])**.** *The eigenvalues of problem (1)-(3) are real and simple, and form an unboundedly increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \quad \text{if} \quad a > 0,$$
$$\lambda_1 < 0 < \lambda_2 < \ldots < \lambda_k < \ldots \quad \text{if} \quad a < 0.$$

Moreover, in the case $a > 0$ *for each* $k \in \mathrm{N}$ *the eigenfunction* $y_k(x)$ *corresponding to the eigenvalue* λ_k *has exactly* $k-1$ *simple zeros in the interval* $(0, 1)$; *in the case* $a < 0$ *for each* $k \in \mathrm{N}$, $k \geq 2$, *the eigenfunction* $y_k(x)$ *has exactly* $k-2$ *simple zeros in the interval* $(0, 1)$, *and the number of zeros of eigenfunction* $y_1(x)$ *can be arbitrary (these zeros are simple).*

It follows from [7, Theorem 5.1] and [2, Theorem 3.1] that the eigenvalues and eigenfunctions of problem (1)-(3) have the following asymptotic formulas.

Theorem 2. *One has the following relations*

$$\sqrt[4]{\lambda_k} = (k - 3/4)\pi + O(1/k), \tag{5}$$

$$y_k(x) = \sin(k - 3/4)\pi x + (-1)^k\, e^{-(k-3/4)\pi(1-x)} + O(1/k), \tag{6}$$

where relation (6) holds uniformly with respect to $x \in [0, 1]$.

As in [1, 3], to study the uniform convergence of spectral expansions of problem (1)-(3), we establish a correspondence between the eigenfunctions of problem (1)-(3) and the following eigenvalue problem

$$\begin{cases} y^{(4)}(x) = \lambda y(x), x \in (0, 1), \\ y(0) = y''(0) = y(1) = y'(1) = 0, \end{cases} \tag{7}$$

the uniform convergence of expansions in terms of the system of eigenfunctions of which has been well studied (see, for example, [1, 3] and references therein). By [4, Theorem 5.4 and 5.5] the eigenvalues of problem (7) are positive and simple, and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^{\infty}$. Moreover, it follows from [2, Theorem 3.1] that the eigenvalues and eigenfunctions of problem (7) have the following asymptiotic formulas:

$$\sqrt[4]{\mu_k} = (k + 1/4)\,\pi + O\left(1/k\right), \tag{8}$$

$$\vartheta_k(x) = \sin(k + 1/4)\,\pi x + (-1)^{k+1}\frac{\sqrt{2}}{2}e^{-(k+1/4)\pi(1-x)} + O\left(1/k\right), \tag{9}$$

where formula (9) holds uniformly with respect to $x \in [0, 1]$.

To establish a correspondence between the eigenfunctions of problem (1)-(3) and problem (7), we need more precise asymptotic formulas for the eigenvalues and eigenfunctions of these problems.

3. Asymptotic formulas for eigenvalues and eigenfunctions of problems (1)-(3) and (7)

Theorem 3. *One has the asymptotic following formulas for the eigenvalues and eigenfunctions of problem (7):*

$$\sqrt[4]{\mu_k} = (k + 1/4)\,\pi + O(e^{-k\pi}), \tag{10}$$

$$\vartheta_k(x) = \sin(k + 1/4)\,\pi x + (-1)^{k+1}\frac{\sqrt{2}}{2}e^{(k+1/4)\pi(x-1)} + O(e^{-k\pi}), \tag{11}$$

where formula (11) holds uniformly with respect to $x \in [0, 1]$.

The proof of this theorem is similar to that of [1, Lemma 3.1].

By (11) we have

$$\|\,\vartheta_k(x)\,\|_2^2 = \int_0^1 |\,\vartheta_k(x)\,|^2 dx = 1 + O\left(\frac{1}{e^{k\pi}}\right). \tag{12}$$

We introduce the following notation:

$$\Phi_k(x) = \frac{\vartheta_k(x)}{\|\,\vartheta_k(x)\,\|_2},\ x \in [0, 1].$$

Then by (12) we get

$$\|\,\Phi_k(x)\,\|_2 = 1.$$

Moreover, by direct calculation from (11) we obtain that the asymptotic formula

$$\Phi_k(x) = \sin(k + 1/4)\,\pi x + (-1)^{k+1}\frac{\sqrt{2}}{2}e^{(k+1/4)\pi(x-1)} + O(e^{-k\pi}) \tag{13}$$

holds uniformly with respect to $x \in [0, 1]$.

Theorem 4. *One has the asymptotic following formulas for the eigenvalues and eigenfunctions of problem (1)-(3):*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{3}{4}\right)\pi + \frac{q_0 + 4/a}{4k\pi} + O\left(\frac{1}{k^2}\right), \tag{14}$$

$$y_k(x) = \sin\left(k - \frac{3}{4}\right)\pi x + (-1)^k \frac{\sqrt{2}}{2} e^{\left(k-\frac{3}{4}\right)\pi(x-1)} - \frac{2/a}{4k\pi}\sin\left(k - \frac{3}{4}\right)\pi x +$$

$$+\frac{(q_0 + 4/a)x - q_0(x) + 2/a}{4k\pi}\cos\left(k - \frac{3}{4}\right)\pi x + \frac{2/a}{4k\pi} e^{-\left(k-\frac{3}{4}\right)\pi x} +$$

$$+(-1)^k \frac{\sqrt{2}}{2} \frac{(q_0 + 4/a)x + 4/a - q_1(x)}{4\rho_k} e^{\left(k-\frac{3}{4}\right)\pi(x-1)} + O\left(\frac{1}{k^2}\right), \tag{15}$$

where formula (15) holds uniformly with respect to $x \in [0, 1]$.

Proof. Suppose that $\lambda = \rho^4$ in (1). It follows from [9, Ch. II, § 4.5, Theorem 1 and § 4.6 formula (27)-(29)], in each subdomain T of the complex $\rho-$plane equation (1) has four linearly independent solutions $\phi_k(x,\rho),\ k = 1, 2, 3, 4,$ regular in ρ (for sufficiently large ρ) and satisfying the following relations

$$\phi_k^{(s)}(x,\rho) = (\rho\omega_k)^s e^{\rho\omega_k x}\left(1 + \frac{q_0(x)}{4\rho\omega_k} + O\left(\frac{1}{\rho^2}\right)\right),\ s = 0, 1, 2, 3,\ k = 1, 2, 3, 4, \tag{16}$$

where $\omega_k,\ k = 1, 2, 3, 4,$ are distinct fourth roots of unity, more precisely, $\omega_1 = -1,\ \omega_2 = -i,\ \omega_3 = i$ and $\omega_4 = 1.$ Then for every $k \in \{1, 2, 3, 4\}$ we have the following relations

$$\phi_k''(0,\rho) = (\rho\omega_k)^2\left(1 + O\left(\frac{1}{\rho^2}\right)\right), \tag{17}$$

$$T\phi_k(0,\rho) - a\rho^4\phi_k(0,\rho) = \phi_k'''(0,\rho) - q(0)\phi_k'(0,\rho) - a\rho^4\phi_k(0,\rho) =$$

$$= \left(\rho^3\omega_k^3 - q(0)\rho\omega_k - a\rho^4\right)\left(1 + O\left(\frac{1}{\rho^2}\right)\right) = -a\rho^4\left(1 - \frac{1}{a\rho\omega_k} + O\left(\frac{1}{\rho^2}\right)\right), \tag{18}$$

$$\phi_k(1,\rho) = e^{\rho\omega_k}\left(1 + \frac{q_0}{4\rho\omega_k} + O\left(\frac{1}{\rho^2}\right)\right), \tag{19}$$

$$\phi_k'(x,\rho) = \rho\omega_k e^{\rho\omega_k}\left(1 + \frac{q_0}{4\rho\omega_k} + O\left(\frac{1}{\rho^2}\right)\right). \tag{20}$$

Hence by the boundary conditions (2), (3) and relations (17)-(20) for the characteristic determinant $\Delta(\lambda)$ we have the following relation

$$\Delta(\lambda)=\begin{vmatrix} \phi_1''(0,\rho) & \phi_2''(0,\rho) & \phi_3''(0,\rho) & \phi_4''(0,\rho) \\ T\phi_1(0,\rho)-a\rho^4\phi_1(0,\rho) & T\phi_2(0,\rho)-a\rho^4\phi_2(0,\rho) & T\phi_3(0,\rho)-a\rho^4\phi_3(0,\rho) & T\phi_4(0,\rho)-a\rho^4\phi_4(0,\rho) \\ \phi_1(1,\rho) & \phi_2(1,\rho) & \phi_3(1,\rho) & \phi_4(1,\rho) \\ \phi_1'(1,\rho) & \phi_2'(1,\rho) & \phi_3'(1,\rho) & \phi_4'(1,\rho) \end{vmatrix}=$$

$$=-a\rho^7[1]\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1+\frac{1}{a\rho} & 1+\frac{1}{ai\rho} & 1-\frac{1}{ai\rho} & 1-\frac{1}{a\rho} \\ e^{-\rho}\left(1-\frac{q_0}{a\rho}\right) & e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & e^{\rho}\left(1+\frac{q_0}{a\rho}\right) \\ -e^{-\rho}\left(1-\frac{q_0}{a\rho}\right) & -ie^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & ie^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & e^{\rho}\left(1+\frac{q_0}{a\rho}\right) \end{vmatrix}=$$

$$=-a\rho^7e^{\rho}\left(1+\frac{q_0}{a\rho}\right)[1]\begin{vmatrix} 1 & -1 & -1 & 0 \\ 1+\frac{1}{a\rho} & 1+\frac{1}{ai\rho} & 1-\frac{1}{ai\rho} & 0 \\ 0 & e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 1 \\ 0 & -ie^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & ie^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 1 \end{vmatrix}=$$

$$=-a\rho^7e^{\rho}\left(1+\frac{q_0}{a\rho}\right)[1]\begin{vmatrix} 1 & -1 & -1 & 0 \\ 1+\frac{1}{a\rho} & 1+\frac{1}{ai\rho} & 1-\frac{1}{ai\rho} & 0 \\ 0 & -ie^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & ie^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 1 \\ 0 & (1+i)e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & (1-i)e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 0 \end{vmatrix}=$$

$$=-a\rho^7e^{\rho}\left(1+\frac{q_0}{a\rho}\right)[1]\begin{vmatrix} 1 & -1 & -1 \\ 1+\frac{1}{a\rho} & 1+\frac{1}{ai\rho} & 1-\frac{1}{ai\rho} \\ 0 & (1+i)e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & (1-i)e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) \end{vmatrix}=$$

$$=-a\rho^7e^{\rho}\left(1+\frac{q_0}{a\rho}\right)[1]\begin{vmatrix} 1+\frac{1}{ai\rho}(1+i) & 1-\frac{1}{ai\rho}(1-i) \\ (1+i)e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & (1-i)e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) \end{vmatrix}=$$

$$=-a\rho^7e^{\rho}\left(1+\frac{q_0}{a\rho}\right)[1]\times\left\{\left(1+\frac{1}{ai\rho}(1+i)\right)(1-i)e^{i\rho}\left(1+\frac{q_0}{4i\rho}\right)-\left(1-\frac{1}{ai\rho}(1-i)\right)(1+i)e^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right)\right\}. \tag{21}$$

It follows from (21) that the eigenvalues of problem (1)-(3) are the roots of the following equation

$$e^{2i\rho}=\frac{1+i}{1-i}\frac{\left(1-\frac{q_0}{4i\rho}\right)\left(1-\frac{1}{ai\rho}(1-i)\right)}{\left(1+\frac{q_0}{4i\rho}\right)\left(1+\frac{1}{ai\rho}(1+i)\right)}=i\left(1-\frac{q_0+4/a}{2i\rho}\right)+O\left(\frac{1}{\rho^2}\right). \tag{22}$$

By asymptotic formula (5) we have

$$\rho_k=\left(k-\frac{3}{4}\right)\pi+\tau_k, \tag{23}$$

where $\tau_k=o(1)$ as $k\to\infty$. Then we get

$$e^{2i\rho_k}=e^{2i\left(k-\frac{3}{4}\right)\pi}(1+2i\tau_k+o(\tau_k^2))=i\,(1+2i\tau_k+o(\tau_k^2)). \tag{24}$$

Hence by (24) we obtain

$$\tau_k=\frac{q_0+4/a}{4\rho_k}+O\left(\frac{1}{\rho_k^2}\right)=\frac{q_0+4/a}{4k\pi}+O\left(\frac{1}{k^2}\right). \tag{25}$$

Therefore, by (25) from (23) we obtain the asymptotic formula (13).

In view of (13) we get the following asymptotic formulas

$$e^{i\rho_k} = (-1)^{k+1}\frac{\sqrt{2}}{2}(1+i)\left(1-\frac{q_0+4/a}{4ik\pi}+O\left(\frac{1}{k^2}\right)\right), \tag{26}$$

$$e^{-i\rho_k} = (-1)^{k+1}\frac{\sqrt{2}}{2}(1-i)\left(1+\frac{q_0+4/a}{4ik\pi}+O\left(\frac{1}{k^2}\right)\right). \tag{27}$$

Using [9, Ch. 2, formula (73)] and formulas (16), (26), (27) for the eigenfunction $y_k(x)$ corresponding to the eigenvalue $\lambda_k = \rho_k^4$ of problem (1)-(3) we havey the following relation

$$y_k(x) = -aC_k\rho_k^6[1]$$

$$\times\begin{vmatrix} e^{-\rho_k x}\left(1-\frac{q_0(x)}{4\rho_k}\right) & e^{-i\rho_k x}\left(1-\frac{q_0(x)}{4i\rho_k}\right) & e^{-\rho_k x}\left(1+\frac{q_0(x)}{4i\rho_k}\right) & e^{\rho_k x}\left(1+\frac{q_0(x)}{4\rho_k}\right) \\ 1 & -1 & -1 & 1 \\ 1+\frac{1}{a\rho_k} & 1+\frac{1}{ai\rho_k} & 1-\frac{1}{ai\rho_k} & 1-\frac{1}{a\rho_k} \\ e^{-\rho}\left(1-\frac{q_0}{4\rho_k}\right) & e^{-i\rho}\left(1-\frac{q_0}{4i\rho_k}\right) & e^{i\rho}\left(1+\frac{q_0}{4i\rho_k}\right) & e^{\rho}\left(1+\frac{q_0}{4\rho_k}\right) \end{vmatrix} =$$

$$= -aC_k\rho_k^6 e^{\rho_k}\left(1+\frac{q_0}{4\rho_k}\right)[1]$$

$$\times\begin{vmatrix} e^{-\rho_k x}\left(1-\frac{q_0(x)}{4\rho_k}\right) & e^{-i\rho_k x}\left(1-\frac{q_0(x)}{4i\rho_k}\right) & e^{-\rho_k x}\left(1+\frac{q_0(x)}{4i\rho_k}\right) & e^{\rho_k(x-1)}\left(1-\frac{q_1(x)}{4\rho_k}\right) \\ 1 & -1 & -1 & 0 \\ 1+\frac{1}{a\rho_k} & 1+\frac{1}{ai\rho_k} & 1-\frac{1}{ai\rho_k} & 0 \\ 0 & e^{-i\rho}\left(1-\frac{q_0}{4i\rho_k}\right) & e^{i\rho}\left(1+\frac{q_0}{4i\rho_k}\right) & 1 \end{vmatrix} =$$

$$=-aC_k\rho_k^6 e^{\rho_k}\left(1+\frac{q_0}{4\rho_k}\right)[1]\times$$

$$\left\{e^{-\rho_k x}\left(1-\frac{q_0(x)}{4\rho_k}\right)\begin{vmatrix} -1 & -1 & 0 \\ 1+\frac{1}{ai\rho_k} & 1-\frac{1}{ai\rho_k} & 0 \\ e^{-i\rho}\left(1-\frac{q_0}{4i\rho_k}\right) & e^{i\rho}\left(1+\frac{q_0}{4i\rho_k}\right) & 1 \end{vmatrix}-\right.$$

$$-e^{-i\rho_k x}\left(1-\frac{q_0(x)}{4i\rho_k}\right)\begin{vmatrix} 1 & -1 & 0 \\ 1+\frac{1}{a\rho_k} & 1-\frac{1}{ai\rho_k} & 0 \\ 0 & e^{i\rho_k}\left(1+\frac{q_0}{4i\rho_k}\right) & 1 \end{vmatrix}+$$

$$+e^{-\rho_k x}\left(1+\frac{q_0(x)}{4i\rho_k}\right)\begin{vmatrix} 1 & -1 & 0 \\ 1+\frac{1}{a\rho_k} & 1+\frac{1}{ai\rho_k} & 0 \\ 0 & e^{-i\rho_k}\left(1-\frac{q_0}{4i\rho_k}\right) & 1 \end{vmatrix}-$$

$$\left.-e^{\rho_k(x-1)}\left(1-\frac{q_1(x)}{4\rho_k}\right)\begin{vmatrix} 1 & -1 & -1 \\ 1+\frac{1}{a\rho_k} & 1+\frac{1}{ai\rho_k} & 1-\frac{1}{ai\rho_k} \\ 0 & e^{-i\rho_k}\left(1-\frac{q_0}{4i\rho_k}\right) & e^{i\rho_k}\left(1+\frac{q_0}{4i\rho_k}\right) \end{vmatrix}\right\}=$$

$$=-4iaC_k\rho_k^6 e^{\rho_k}\left(1+\frac{q_0}{4\rho_k}\right)\left\{\sin\rho_k x+(-1)^k\frac{\sqrt{2}}{2}e^{\rho_k(x-1)}-\frac{2/a}{4\rho_k}\sin\rho_k x-\right.$$

$$-\frac{q_0(x)+2/a}{4\rho_k}\cos\rho_k x+\frac{2/a}{4\rho_k}e^{-\rho_k x}+$$

$$\left.+(-1)^k\frac{\sqrt{2}}{2}\frac{4/a-q_1(x)}{4\rho_k}e^{\rho_k(x-1)}+O\left(\frac{1}{\rho_k^2}\right)\right\}. \qquad (28)$$

We choose the constant C_k such that $-4iaC_k\rho_k^6e^{\rho_k}\left(1+\frac{q_0}{4\rho_k}\right)=1$. Then, by (13), it follows from (28) that

$$y_k(x)=\sin\left(k-\frac{3}{4}\right)\pi x+(-1)^k\frac{\sqrt{2}}{2}e^{\left(k-\frac{3}{4}\right)\pi(x-1)}-\frac{2/a}{4k\pi}\sin\left(k-\frac{3}{4}\right)\pi x+$$

$$+\frac{(q_0+4/a)x-q_0(x)+2/a}{4k\pi}\cos\left(k-\frac{3}{4}\right)\pi x+\frac{2/a}{4k\pi}e^{-\left(k-\frac{3}{4}\right)\pi x}+$$

$$+(-1)^k\frac{\sqrt{2}}{2}\frac{(q_0+4/a)x+4/a-q_1(x)}{4k\pi}e^{\left(k-\frac{3}{4}\right)\pi(x-1)}+O\left(\frac{1}{k^2}\right). \qquad (29)$$

The proof of this theorem is complete.

4. Uniform convergence of Fourier series expansions of the system of eigenfunctions of problem (1)-(3)

By the asymptotic formulas (13) and (29) for $k\geq 2$ we have the following relation

$$y_k(x)=\Phi_{k-1}(x)-\frac{2/a}{4k\pi}\sin\left(k-\frac{3}{4}\right)\pi x+$$

$$+\frac{(q_0+4/a)x-q_0(x)+2/a}{4k\pi}\cos\left(k-\frac{3}{4}\right)\pi x+\frac{2/a}{4k\pi}e^{-\left(k-\frac{3}{4}\right)\pi x}+$$

$$+(-1)^k\frac{\sqrt{2}}{2}\frac{(q_0+4/a)x+4/a-q_1(x)}{4k\pi}e^{\left(k-\frac{3}{4}\right)\pi(x-1)}+O\left(\frac{1}{k^2}\right). \qquad (30)$$

Moreover, from these relations we obtain

$$\Phi_k(0)=O\left(\frac{1}{e^{k\pi}}\right),\ y_k(0)=\frac{1}{ak\pi}+O\left(\frac{1}{k^2}\right),\ y_k^2(0)=O\left(\frac{1}{k^2}\right). \qquad (31)$$

Let r be an arbitrary fixed natural number. Then by Theorems 1 and 2 it follows from [7, Theorem 8.1] and [2, Theorem 6.2] that the system $\{y_k(x)\}_{k=1,k\neq r}^{\infty}$ of eigenfunctions of problem (1)-(3) forms a basis in the space $L_p(0,1),\ 1<p<\infty,$ which is an unconditional basis in the space $L_2(0,1)$. In this case the system $\{u_k(x)\}_{k=1,k\neq r}^{\infty}$ adjoint to the system $\{y_k(x)\}_{k=1,k\neq r}^{\infty}$ is defined by the formula

$$u_k(x) = \delta_k^{-1}\left\{ y_k(x) - \frac{y_k(0)}{y_r(0)}\, y_r(x) \right\}, \tag{32}$$

where

$$\delta_k = \| y_k \|_2^2 + a y_k^2(0) \neq 0,\ k \in \mathrm{N}. \tag{33}$$

By following the arguments in pp. 282-284 [8] we can show that

$$\| y_k(x) \|_2^2 = 1 + O\left(\frac{1}{k^2} \right). \tag{34}$$

By (31), (33) and (34) from (32) we get

$$u_k(x) = y_k(x) - \frac{a y_k(0)}{a y_r(0)}\, y_r(x) + O\left(\frac{1}{k^2} \right). \tag{36}$$

By the above arguments the Fourier series

$$f(x) = \sum_{k=1,k\neq r}^{\infty} (f, u_k)_{L_2}\, y_k(x), \tag{37}$$

where $(f, u_k)_{L_2} = \int_0^1 f(x) y_k(x) dx,$ of continuous function $f(x)$ in the system $\{y_k(x)\}_{k=1,k\neq r}^{\infty}$ of eigenfunctions of problem (1)-(3) converges in $L_p(0,1),\ 1<p<\infty.$

The main result of this paper is the following theorem.

Theorem 5. Let r be an arbitrarily fixed positive integer, $f(x)$ is a continuous function on $[0, 1]$ and has a uniformly convergent Fourier series expansion in the system $\{\Phi_k(x)\}_{k=1}^{\infty}$ on $[0,1]$. If $(f, y_r)_{L_2} \neq 0,$ then the series (37) is uniformly convergent on any interval $[\chi, 1],\ 0<\chi<1,$ if $(f, y_r)_{L_2} = 0,$ then the series (37) is uniformly convergent on the interval $[0, 1]$.

The proof of this theorem is similar to that of [1, Theorem 5.1] with the use of (30)-(36).

References

[1] Abdullayeva KF, Aliyev ZS, Kerimov NB. On the uniform convergence of Fourier series expansions in the system of eigenfunctions of the equation of a vibrating rod at one end of which the mass is concentrated, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **2020**, v. 46 (2), p. 226-242.

[2] Aliev ZS. Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition. Diff. Equ. **2011**, v. 47 (6), p. 766-777.

[3] Aliyev ZS, Kerimov NB, Mehrabov VA. Convergence of eigenfunction expansions for a boundary value problem with spectral parameter in the boundary conditions. II. Diff. Equ. **2020,** v. 56 (3), p. 277-289.

[4] Banks DO, Kurowski GJ. A Prüfer transformation for the equation of a vibrating beam. Trans. Amer. Math. Soc. **1974**, v. 199, p. 203-222.

[5] Bolotin B.B. Vibrations in Technique: Handbook in 6 Volumes, The Vibrations of Linear Systems, I. Engineering Industry, Moscow, **1978**.

[6] Jafarova MS. Basis property of root functions of some eigenvalue problem with spectral parameter contained in the boundary condition, BSUJMCS, **2024**, v. 1 (3), p. 25-33.

[7] Kerimov N.B., Aliev ZS. Basis properties of a spectral problem with spectral parameter in the boundary condition. Sb. Math. **2006,** v. 197 (10), p. 65-86.

[8] Kerimov NB, Aliev ZS. On the basis property of the system of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition. Differ. Equ. **2007**, v. 43 (7), p. 905-915.

[9] Naımark MA. Linear differential operators, Frederick Ungar, New York, 1968

[10] Roseau M. Vibrations in Mechanical Systems, Analytical Methods and Applications. Springer, Berlin, **1987**.