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EIGENVALUES AND EIGENFUNCTIONS OF A DIFFERENTIAL OPERATOR WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this work we study the second order differential operator with integral boundary conditions. Under weaker than previously known conditions on the functions $\varphi_\nu(x),\ \nu=1,2,$ asymptotic formulas for eigenvalues and Eigen functions are found functions, an estimate of the resolvent was obtained and theorem on completeness and minimality of eigenfunctions in some subspace of space $L_p(0,1),\ 1<p<\infty,$ codimension 2.

Keywords: second order differential operator, integral boundary conditions, completeness, minimality, eigenvalues, eigenfunctions

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1. Introduction.

Consider the linear differential expression

$$l(y)=-y''+q(x)y,\ \ x\in(0,1) \tag{1}$$

and boundary conditions

$$U_1(y)=U_2(y)=0 \tag{2}$$

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where $q(x)-$ is a complex-valued function summable on $[0,1]$ and $U_1(y)$ and $U_2(y)$ – are the corresponding boundary forms. Differential expression (1) and boundary conditions (2) generate a differential operator L with a domain of definition $D(L)$ in some functional space X. We will be interested in the problem of the behavior of eigenvalues and eigenfunctions of this differential operator. Such a problem in the case of regular boundary conditions $U_v(y)=0,\ v=1,2,$ has been studied quite well (see [1,2] and the bibliography there). The case of irregular, as well as more general regular boundary conditions, when the boundary conditions contain some integrals of the function $y(x)$ and its derivatives, was considered in [3-6]. In these works, the spectral properties of the corresponding operator were studied (spectrality, eigenfunctions, conjugate problem and mainly in the space $L_2(0,1)$). Let us also note the works [8-10], where similar problems were studied in the spaces $L_p(0,1)$. However, as a rule, boundary forms generated an unbounded functional in the space under consideration, and in this case the operator has a dense domain of definition, which made it possible to construct a conjugate operator or assume the regularity of boundary conditions [1,2,4]. Here we will consider integral boundary conditions

$$U_v(y)=\int_0^1 \varphi_v(x)y(x)dx=0, \quad v=1,2, \tag{3}$$

where $\varphi_v(x)$ - are given linearly independent functions belonging to the space $L_q(0,1), \frac{1}{p}+\frac{1}{q}=1.$ These conditions are not regular in the sense of Birkhoff [1], and there is no corresponding conjugate operator for them. Such conditions were used for other purposes in [6, 7]. In [11, 12], problem (1), (3) was studied under more stringent conditions on the functions $q(x)$ and $\varphi_v(x)$, where the asymptotic behavior of eigenvalues and eigenfunctions was found, and the theorem on the Riesz basis property of a system of eigenfunctions in $L_2(0,1)$. Note that differential equations with nonlocal conditions of integral form have interesting applications in mechanics [13] and in the theory of diffusion processes

[14].

1. Basic assumptions.

Let us introduce in the space $L_p(0,1), 1 < p < \infty,$ a differential operator $L,$ corresponding to the differential expression $l(y)$ with the domain of definition $D(L) = \{y(x) \in W_p^2(0,1),\, l(y) \in L_p(0,1);\, U_v(y) = 0,\, v = 1,2\}$ and consider the problem of the eigenvalues of this operator

$$Ly = \lambda y. \tag{4}$$

Let's put $\lambda = -\rho^2$. Equation (1) has [1, pg. 58] fundamental system of solutions

$$y_1(x,\rho) = e^{i\rho x}(1 + r_1(x,\rho),\; y_2(x,\rho) = e^{-i\rho x}(1 + r_2(x,\rho)), \tag{5}$$

where the functions $r_i(x,\rho)$ are continuous even for large values of $|\rho|$ the estimate $|r_i(x,\rho)| \le \frac{c_i}{|\rho|},\, i = 1,2$ is satisfied, uniformly in $x \in [0,1]$.

Regarding the functions $\varphi_v(x),\, v = 1,2,$ we will additionally assume that in some strip $|\operatorname{Im}\rho| \le h,$ for some $h > 0$ the following relations are satisfied:

$$\begin{gathered}\int_0^1 \varphi_v(x) e^{i\rho x} dx = \frac{1}{i\rho}\left(\beta_v e^{i\rho} - \alpha_v\right) + o\left(\frac{1}{\rho}\right),\\ \int_0^1 \varphi_v(x) e^{-i\rho x} dx = \frac{1}{-ip}\left(\beta_v e^{-i\rho} - \alpha_v\right) + o\left(\frac{1}{\rho}\right),\end{gathered} \tag{6}$$

where $|a_v| + |b_v| \ne 0$. These relations are satisfied, for example, if the functions $\varphi_v(x),\, q(x)$, are smooth in a small neighborhood of the points $x = 0$ and $x = 1$. Moreover, in equalities (6) $\alpha_v = \varphi_v(0),\, \beta_v = \varphi_v(1)$ (one of the numbers α_v, β_v can become zero). For functions $\varphi_v(x) \in W_1^1(0,1)$ such expansions are obtained using integration by parts and from the Riemann theorem. In addition, from (5) and (6) it follows that under the same assumptions the relations are also satisfied

$$\int_0^1 \varphi_v(x) y_1(x,\rho) dx = \frac{1}{i\rho}\left(\beta_v e^{i\rho} - \alpha_v\right) + \frac{r_{v1}(\rho)}{\rho},$$

$$\int_0^1 \varphi_v(x)y_2(x,\rho)dx = \frac{1}{-i\rho}\left(\beta_v e^{-i\rho} - \alpha_v\right) + \frac{r_{v2}(\rho)}{\rho}, \qquad (7)$$

where for functions $r_{vi}(\rho)$ for large values of $|\rho|$ and $|\operatorname{Im}\rho| \le h$ the estimate $r_{vi}(\rho) = o(1)$ is satisfied. In what follows we will assume that the condition $\alpha_1\beta_2 - \alpha_2\beta_1 \ne 0$ is satisfied.

2. Asymptotic behavior of eigenvalues.

The main result of this point is following theorem:

Theorem 1. *Let the function $q(x)$ be summable, and the functions $\varphi_v(x),\ v = 1,2,$ be summable with degree $q, \frac{1}{p} + \frac{1}{q} = 1,$ on the interval $[0,1]$ and for functions $\varphi_v(x)$ the asymptotic representations (6), (7) take place. Then the following asymptotic formula for the eigenvalues of the operator L is valid:*

$$\rho_k = \pi k + o(1)$$

Proof. To find the eigenvalues of the operator L, consider the determinant

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix},$$

where $y_1(x)$ and $y_2(x)$ - are the fundamental system of solutions from (5). The determinant $\Delta(\rho)$ is divided into the sum

$$\Delta(\rho) = \Delta_0(\rho) + \Delta_1(\rho) + \Delta_2(\rho) + \Delta_3(\rho), \qquad (8)$$

where

$$\Delta_0(\rho) = \begin{vmatrix} \int_0^1 \varphi_1(x)e^{i\rho x}dx & \int_0^1 \varphi_1(x)e^{-i\rho x}dx \\ \int_0^1 \varphi_2(x)e^{i\rho x}dx & \int_0^1 \varphi_2(x)e^{-i\rho x}dx \end{vmatrix}, \qquad (9)$$

$\Delta_1(\rho)$ is obtained from $\Delta_0(\rho)$ by replacing the second row with the elements

$\int_0^1 \varphi_2(x)e^{i\rho x} r_1(x,\rho)dx,$ $\int_0^1 \varphi_2(x)e^{-i\rho x} r_2(x,\rho)dx,$ and $\Delta_2(\rho)-$ from $\Delta_0(\rho)$ by replacing the first row with the elements $\int_0^1 \varphi_1(x)e^{i\rho x} r_1(x,\rho)dx,$ $\int_0^1 \varphi_1(x)e^{-i\rho x} r_2(x,\rho)dx,$ finally, $\Delta_3(\rho)$ - replacing both lines with the specified elements (the first with c $\varphi_1(x),$ the second with $\varphi_2(x)$). Let's consider the determinant $\Delta_1(\rho)$ By virtue of formulas (6), (7) we have

$$\Delta_1(\rho) = \frac{R_1(\rho)}{\rho^2},\ \Delta_2(\rho) = \frac{R_2(\rho)}{\rho^2},\ \Delta_3(\rho) = \frac{R_3(\rho)}{\rho^3}$$

where $R_i(\rho) = o(1),\ i = 1,2,3,$ for $\rho \to \infty$ and $|\mathrm{Im}\,\rho| \le h$. Thus, in expansion (4) the main role as $\rho \to \infty$ is played by the term $\Delta_0(\rho)$, therefore, taking into account formulas (8), (9) we have

$$\Delta(\rho) = \frac{1}{\rho^2}\begin{vmatrix} \beta_1 e^{i\rho} - \alpha_1 & \beta_1 e^{-i\rho} - \alpha_1 \\ \beta_2 e^{i\rho} - \alpha_2 & \beta_2 e^{-i\rho} - \alpha_2 \end{vmatrix} + \frac{R(\rho)}{\rho^2}, \tag{10}$$

where $R(\rho) = o(1)$ for $\rho \to \infty$ and $|\mathrm{Im}\,\rho| \le h$. From equality (5) it follows that

$$\Delta(\rho) = \frac{1}{\rho^2}(\alpha_1\beta_2 - \alpha_2\beta_1)(e^{i\rho} - e^{-i\rho}) + \frac{R(\rho)}{\rho^2},$$

and the roots of the equation $\Delta(\rho) = 0$ (see. [1, pg. 77]) form the sequence

$$\rho_k = \pi k + o(1) \text{ for } \alpha_1\beta_2 - \alpha_2\beta_1 = 0,\ k \in Z \setminus \{0\}.$$

3. Asymptotics of eigenfunctions.

Let's move on to finding the eigenfunctions of the operator L. As usual [1, p. 84], they are searched in the form

$$y_k(x) = c_k \begin{vmatrix} y_1(x,\rho_k) & y_2(x,\rho_k) \\ U_2(y_1) & U_2(y_2) \end{vmatrix} =$$

$$= c_k \begin{vmatrix} e^{i\rho_k x}(1 + r_1(x,\rho_k)) & e^{-i\rho_k x}(1 + r_2(x,\rho_k)) \\ \int_0^1 \varphi_2(x) e^{i\rho_k x}(1 + r_1(x,\rho_k))dx & \int_0^1 \varphi_2(x) e^{i\rho_k x}(1 + r_2(x,\rho_k))dx \end{vmatrix} \tag{11}$$

where c_k is some normalizing factor to be determined. We have $\rho_k = \pi k + o(1)$ and $r_1(x,\rho_k) = o(1),\ r_2(x,\rho_k) = o(1)$. Substituting these expressions into the determinant (11) and using formulas (6), we obtain

$$y_k(x) = c_k \left(\begin{vmatrix} e^{i\rho_k x} & e^{-i\rho_k x} \\ \frac{1}{i\rho_k}(\beta_2 e^{i\rho_k} - \alpha_2) & \frac{1}{-i\rho_k}(\beta_2 e^{-i\rho_k} - \alpha_2) \end{vmatrix} + o\left(\frac{1}{\rho_k}\right) \right). \tag{12}$$

Further, taking into account that $e^{i\rho_k} = (-1)^k + o(1),\ e^{-i\rho_k} = (-1)^k + o(1),$ $e^{i\rho_k x} = e^{i\pi kx} + o(1),\ e^{-i\rho_k x} = e^{-i\pi kx} + o(1),$ from (12) we obtain

$$y_k(x) = c_k \frac{1}{i\pi k}\left(\left(\alpha_2 - (-1)^k \beta_2\right)\left(e^{i\pi kx} + e^{-i\pi kx}\right) + o\left(\frac{1}{k}\right)\right).$$

Now choosing $c_k = \frac{i\pi k}{2(\alpha_2 - (-1)^k \beta_2)}$, from the last equality we get

$$y_k(x) = cox\pi kx + o(1).$$

Thus it is proven.

Theorem 2. *Under the conditions of Theorem 1, the following asymptotic formulas hold for the eigenfunctions of the operator* $L : y_k(x) = cox\pi kx + o(1)$. .

4. Completeness of eigenfunctions.

The operator L constructed in paragraph 2 does not have a dense domain of definition in the space $L_p(0,1)$ and therefore the eigenfunctions of the operator L cannot be complete in this space. To eliminate this drawback, consider the operator L not in the entire space $L_p(0,1)$,but in its closed subspace

$$X_p = \{ f(x) \in L_p(0,1) : U_\nu(f) = 0,\ \nu = 1,2 \}.$$

It is obvious that $codim\ X_p = 2$ Let us define the operator L in the space X_p as follows:

$D(L) = \{y \in W_p^2(0,1) \cap X_p : l(y) \in X_p\}$ and for $y \in D(L) : Ly = l(y)$.

The operator L thus defined has an everywhere dense domain of definition in X_p To study the question of completeness of eigenfunctions of the operator L in the space X_p we construct and estimate the resolvent of the operator L It is known (see [1]) that the Green's function of the operator $L - \rho^2 I$ has the form

$$G(x,\xi,\rho) = \frac{1}{\Delta(\rho)} \begin{vmatrix} g(x,\xi,\rho) & y_1(x,\rho) & y_2(x,\rho) \\ U_1(g) & U_1(y_1) & U_1(y_2) \\ U_2(g) & U_2(y_1) & U_2(y_2) \end{vmatrix} \tag{13}$$

Where

$$g(x,\xi,\rho) = \begin{cases} -y_1(x,\rho) z_1(\xi,\rho), & x \geq \xi, \\ y_2(x,\rho) z_2(\xi,\rho), & x < \xi, \end{cases}$$

$$z_1(\xi,\rho) = \frac{y_2(\xi,\rho)}{W(\rho)}, \quad z_2(\xi,\rho) = \frac{y_1(\xi,\rho)}{W(\rho)}, \quad W(\rho) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Consider in the complex ρ -plane the region $\Omega_\delta = \{\rho : |\rho - \rho_k| \geq \delta\}$, where $\{\rho_k\}$- is the set of zeros of the function $\Delta(\rho)$ Let K_δ denote the region of the complex $\lambda -$plane, which is the image of Ω_δ under the mapping $\lambda = \rho^2$.

Theorem 3. For the resolvent $R_\lambda(L) = (L - \lambda I)^{-1}$ of the operator L in the domain K_δ for large values of $|\lambda|$ the following estimate is correct:

$$\|R_\lambda(L)\| \leq \frac{c}{|\lambda|^{\frac{1}{2}}}. \tag{14}$$

Proof. It is known [1] that for the derivatives of the functions $y_1(x,\rho)$ and $y_2(x,\rho)$ in the region $T = c + S_0$, where $S_0 = \{\rho : \operatorname{Im}\rho \geq 0, \operatorname{Re}\rho \geq 0\}$ asymptotic estimates are valid

$$y_1'(x,\rho)=i\rho e^{i\rho x}(1+r_3(x,\rho)),\ \ y_2'(x,\rho)=-i\rho e^{-i\rho x}(1+r_4(x,\rho)),$$

where $\left|r_i(x,\rho)\right|\le\frac{c_i}{|\rho|},\ i=3,4,$ uniformly along $x\in[0,1]$. From the last relations, taking into account (6) for the Wronskian $W(\rho)$, we have

$$W(\rho)=\begin{vmatrix} e^{i\rho x}(1+r_1(x,\rho)) & e^{-i\rho x}(1+r_2(x,\rho)) \\ ie^{i\rho x}(1+r_3(x,\rho)) & -ie^{-i\rho x}(1+r_3(x,\rho)) \end{vmatrix}=-2i\rho\left(1+0\left(\frac{1}{\rho}\right)\right).$$

From here, for the functions $z_1(\xi,\rho)$ and $z_2(\xi,\rho)$ we obtain

$$z_1(\xi,\rho)=-\frac{1}{2i\rho}e^{-i\rho\xi}\left(1+0\left(\frac{1}{\rho}\right)\right),\ z_2(\xi,\rho)=\frac{1}{2i\rho}e^{i\rho\xi}\left(1+0\left(\frac{1}{\rho}\right)\right).$$

Therefore for $g(x,\xi,\rho)$ we have

$$g(x,\xi,\rho)=\begin{cases} \frac{1}{2i\rho}e^{i\rho(x-\xi)}\left(1+0\left(\frac{1}{\rho}\right)\right), & x\ge\xi, \\ -\frac{1}{2i\rho}e^{-i\rho(x-\xi)}\left(1+0\left(\frac{1}{\rho}\right)\right), & x<\xi. \end{cases}$$

Taking into account formulas (6), as well as the last relations, we estimate $U_1(g)$ and $U_2(g)$:

$$U_1(g)=\int_0^1\varphi_1(x)g(x,\xi,\rho)dx=-\frac{1}{2i\rho}e^{i\rho\xi}\int_0^{\xi}\varphi_1(x)e^{-i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx+$$

$$+\frac{1}{2i\rho}e^{-i\rho\xi}\int_{\xi}^{0}\varphi_1(x)e^{i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx=\frac{1}{2\rho^2}\left(a_1e^{i\rho\xi}-b_1^{\ i\rho(1-\xi)}+o\left(\frac{1}{\rho^2}\right)\right),$$

$$U_2(g)=\int_0^1\varphi_2(x)g(x,\xi,\rho)dx=-\frac{1}{2i\rho}e^{i\rho\xi}\int_0^{\xi}\varphi_2(x)e^{-i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx+$$

$$+\frac{1}{2i\rho}e^{-i\rho\xi}\int_{\xi}^{0}\varphi_2(x)e^{i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx=\frac{1}{2\rho^2}\left(a_2e^{i\rho\xi}-b_2^{\ i\rho(1-\xi)}+o\left(\frac{1}{\rho^2}\right)\right).$$

Finally, we replace all the functions included in (13) with their asymptotic expressions. Then for $x \geq \xi$ we have

$$G(x,\xi,\rho) = \frac{1}{\Delta(\rho)}\begin{vmatrix} g(x,\xi,\rho) & y_1(x,\rho) & y_2(x,\rho) \\ U_1(g) & U_1(y_1) & U_1(y_2) \\ U_2(g) & U_2(y_1) & U_2(y_2) \end{vmatrix} =$$

$$= \frac{1}{2\rho(\alpha_1\beta_2 - \alpha_2\beta_1)(e^{i\rho} - e^{-i\rho})[1]} \times$$

$$\times \begin{vmatrix} e^{i\rho(1-\xi)}[1] & e^{i\rho x}[1] & e^{i\rho x}[1] \\ (\alpha_1 e^{i\rho\xi} - \beta_1 e^{i\rho(1-\xi)})[1] & (\alpha_1 - \beta_1 e^{i\rho})[1] & (\beta_1 e^{-i\rho} - \alpha_1)[1] \\ (\alpha_2 e^{i\rho\xi} - \beta_2 e^{i\rho(1-\xi)})[1] & (\alpha_2 - \beta_2 e^{i\rho})[1] & (\beta_2 e^{-i\rho} - \alpha_2)[1] \end{vmatrix} =$$

$$\frac{1}{2\rho(\alpha_1\beta_2 - \alpha_2\beta_1)(e^{2ip} - 1)[1]} \times$$

$$\times \begin{vmatrix} e^{i\rho(x-\xi)}[1] & e^{i\rho x}[1] & e^{i\rho(1-x)}[1] \\ (\alpha_1 e^{i\rho\xi} - \beta_1 e^{i\rho(1-\xi)})[1] & (\alpha_1 - \beta_1 e^{i\rho})[1] & (\beta_1 - \alpha_1 e^{i\rho})[1] \\ (\alpha_2 e^{i\rho\xi} - \beta_2 e^{i\rho(1-\xi)})[1] & (\alpha_2 - \beta_2 e^{i\rho})[1] & (\beta_2 - \alpha_2 e^{i\rho})[1] \end{vmatrix}.$$

Here we use the notation $[a] = a + o(1)$. Note that if p belongs to the domain, $T \cap \Omega_\delta$ then $\mathrm{Re}(i\rho) \leq 0$ and therefore there is a number $m_\delta > 0$ such that. $\left|\left(e^{2i\rho} - 1\right)[1]\right| \geq m_\delta$. In addition, in the last determinant all components are limited, since all exponents present there in the indicator have a negative real part. From what has been said it immediately follows that the estimate

$$\left|G(x,\xi,\rho)\right| \leq \frac{c}{|\rho|}, \quad \rho \in T \cap \Omega_\delta, x, \xi \in [0,1]$$

from which we directly obtain (14). The case $x < \xi$ is studied similarly.

The main result of this point is

Theorem 4. The eigen and associated functions of the operator L form a complete and minimal system in the space X_p, $1 \leq p < \infty$.

Proof. Let $\hat{L}$ denote the maximal operator generated in the space $L_p(0,1)$ by differential expression (1), i.e. $D(\hat{L}) = \{y \in W_p^2(0,1),\ l(y) \in L_p(0,1)\}$. Obviously, $\hat{L}$ is a bounded operator acting from $W_p^2(0,1)$ to $L_p(0,1)$. Then from the complete continuity of the embedding operator $W_p^2(0,1)$ into $L_p(0,1)$ and Theorem 6.29 from [15, p. 187] on an operator with a compact resolvent it follows that the operator L also has a compact resolvent. Consequently, the system of eigen and associated functions $\{y_k\}$ is minimal in X_p, since it has a biorthogonal system $\{z_k\}$, which is a system of eigen and associated functions of the conjugate operator L^*.

Let us prove the completeness of the system $\{y_k\}$. Let it not be complete in X_p. Then there exists an element $g \in X_p^*$, , such that $\forall k : \langle y_k, g \rangle = 0.$ From this we obtain that $R_\lambda(L^*)g$ is an entire function of λ. On the other hand, from estimate (14) it follows that $\|R_\lambda(L^*)\| = \|R_\lambda(L)\| = 0\left(\frac{1}{\lambda^{\frac{1}{2}}}\right)$ and according to Liouville's theorem, this is possible only in the case $R_\lambda(L^*)g = const.$ Then, differentiating the latter, we obtain $\frac{d}{d\lambda} R_\lambda(L^*)g = R_\lambda^2(L^*)g = 0$ Hence, from the uniqueness of $R_\lambda(L^*)g$ and therefore $R_\lambda^2(L^*)$, we find that $g = 0$, which completes the proof of the theorem.

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