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ON A NEW CLASS OF REAMANNIAN METRICS ON THE COFRAME BUNDLE

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Abstract

In this paper we introduce a new class of Riemannian metrics on the coframe bundle over a Riemannian manifold (M,g) and investigate the Levi-Civita connection of these metrics. Also we calculate the particular values of components of Levi-Civita connection for different indices.

Keywords: *Sasaki metric, coframe bundle, horizontal lift, adapted frame, Levi-Civita connection*

Mathematics Subject Classification *(2020): 57R30, 53C12*

1. Introduction

The geometries of the tangent, cotangent, linear frame, linear coframe and tensor bundles equipped with Sasaki type metrics has been studied by many authors such as Sasaki S. [13], Yano K. and Ishihara S. [14], Kowalski O. and Sekizawa M. [6], Salimov A.A., Ağca F., Akbulut K., Gezer A., Fattayev H.D. (see [1], [3], [7], [11], [12]), Cordero L. and Leon de M. [2], Zagane M. [15]. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on above mentioned bundles (see, for example, [3], [4]). Salimov A.A. and Fattayev H.D. have introduce the notions of homogeneous deformation of the Sasaki

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metric [10] and Cheeger-Gromoll metric (see [8], [9]) on the coframe bundle over a Riemannian manifold. The main idea in this paper in the new modification of the Sasaki metric on the coframe bundle. First we introduce a new class of Riemannian metrics, noted ${}^{f}g$ on the coframe bundle $F^{*}(M)$ over an $n-$ dimensional Riemannian manifold (M,g), where f is a strictly positive smooth function on M. Then, we investigate the properties of Levi-Civita connection ${}^{f}\nabla$ (Theorem 4.3) of the metric ${}^{f}g$ and we calculate the values of components of ${}^{f}\nabla$ (Theorem 5.1). All manifolds, tensor fields and connections in the present paper are always assumed to be differentiable of class C^{∞}. We denote by $\Im_{q}^{p}(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_{q}^{p}(F^{*}(M))$ the corresponding set on the coframe bundle $F^{*}(M)$. The Einstein summation convention is used.

2. Preliminaries

Let (M,g) be an $n-$dimensional Riemannian manifold. The linear coframe bundle $F^{*}(M)$ over M consists of all pairs (x,u^{*}), where x is a point of M and u^{*} is a basis (coframe) for the cotangent space $T_{x}^{*}M$ of M at x [3]. We denote by π the natural projection of $F^{*}(M)$ to M defined by $\pi(x,u^{*})=x$. If $(U;x^{1},x^{2},...,x^{n})$ is a system of local coordinates in M, then a coframe $u^{*}=(X^{\alpha})=(X^{1},X^{2},...,X^{n})$ for $T_{x}^{*}M$ can be expressed uniquely in the form $X^{\alpha}=X_{i}^{\alpha}(dx^{i})_{x}$. From mentioned above it follows that

$$\left(\pi^{-1}(U);x^{1},x^{2},...,x^{n},X_{1}^{1},X_{2}^{1},...,X_{n}^{n}\right)$$

is a system of local coordinates in $F^{*}(M)$, that is $F^{*}(M)$ is a $C^{\infty}-$ manifold of dimension $n+n^{2}$. We note that indices $i,j,k,...,\alpha,\beta,\gamma,...$ have range in $\{1,2,...,n\}$, while indices $A,B,C,...$ have range in $\{1,...,n,n+1,...,n+n^{2}\}$. We put $h_{\alpha}=\alpha\cdot n+h$. Obviously that indices $h_{\alpha},k_{\beta},l_{\gamma},...$ have range in

$\{n+1, n+2, ..., n+n^2\}$. Let ∇ be a symmetric linear connection on M with components Γ_{ij}^k. Then the tangent space $T_{(x,u^*)}(F^*(M))$ of $F^*(M)$ at $(x,u^*) \in F^*(M)$ splits into the horizontal and vertical subspaces with respect to ∇:

$$T_{(x,u^*)}(F^*(M)) = H_{(x,u^*)}(F^*(M)) \oplus V_{(x,u^*)}(F^*(M)). \qquad (1)$$

From (1) it follows that for every $X \in \Im_0^1(F^*(M))$ is obtained unique decomposing $X = hX + vX$, where $hX \in H(F^*(M))$, $vX \in V(F^*(M))$. $H(F^*(M))$ and $V(F^*(M))$ the horizontal and vertical distributions for $F^*(M)$, respectively. Now we define naturally n different vertical lifts of $1-$ form $\omega \in \Im_1^0(M)$. If Y be a vector field on M, i.e. $Y \in \Im_0^1(M)$, then $i^\mu Y$ are functions on $F^*(M)$ defined by $(i^\mu Y)(x,u^*) = X^\mu(Y)$ for all $(x,u^*) = (x, X^1, X^2, ..., X^n) \in F^*(M)$, where $\mu = 1,2,...,n$. The vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ are the n vector fields such that

$${}^{V_\lambda}\omega(i^\mu Y) = \omega(Y)\delta_\mu^\lambda$$

hold for all vector fields Y on M, where $\lambda, \mu = 1,2,...,n$ and δ_μ^λ denote the Kronecker's delta. The vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ have the components

$${}^{V_\lambda}\omega = \begin{pmatrix} {}^{V_\lambda}\omega^k \\ {}^{V_\lambda}\omega^{k_\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_k \delta_\mu^\lambda \end{pmatrix} \qquad (2)$$

with respect to the induced coordinates (x^i, X_i^α) in $F^*(M)$ (see [10]).

Let $V \in \Im_0^1(M)$. The complete lift ${}^C V \in \Im_0^1(F^*(M))$ of V to the linear coframe bundle $F^*(M)$ is defined by

$${}^C V(i^\mu Y) = i^\mu(L_V Y) = X_m^\mu (L_V Y)^m$$

for all vector fields $Y \in \Im_0^1(M)$, where L_V be the Lie derivation with respect to V. The complete lift ${}^C V$ has the components

$$^{C}V = \begin{pmatrix} ^{C}V^{k} \\ ^{C}V^{k_{\mu}} \end{pmatrix} = \begin{pmatrix} V^{k} \\ -X_{m}^{\mu}\partial_{k}V^{m} \end{pmatrix} \qquad (3)$$

with respect to the induced coordinates (x^{i}, X_{i}^{α}) in $F^{*}(M)$, (see [3])..

The horizontal lift $^{H}V \in \Im_{0}^{1}(F^{*}(M))$ of V to the linear coframe bundle $F^{*}(M)$ is defined by

$$^{H}V(i^{\mu}Y) = i^{\mu}(\nabla_{V}Y) = X_{m}^{\mu}(\nabla_{V}Y)^{m}$$

for all vector fields $Y \in \Im_{0}^{1}(M)$, where ∇_{V} be the covariant derivative with respect to V. The horizontal lift ^{H}V has the components

$$^{H}V = \begin{pmatrix} ^{H}V^{k} \\ ^{H}V^{k_{\mu}} \end{pmatrix} = \begin{pmatrix} V^{k} \\ X_{m}^{\mu}\Gamma_{lk}^{m}V^{l} \end{pmatrix} \qquad (4)$$

with respect to the induced coordinates (x^{i}, X_{i}^{α}) in $F^{*}(M)$, where Γ_{ij}^{k} are the components of Levi-Civita connection on M [10].

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned} &[^{V_{\beta}}\omega, {}^{V_{\gamma}}\theta] = 0, \\ &[^{H}X, {}^{V_{\gamma}}\theta] = {}^{V_{\gamma}}(\nabla_{X}\theta), \\ &[^{H}X, {}^{H}Y] = {}^{H}[X,Y] + \sum_{\sigma=1}^{n} {}^{V_{\sigma}}(X^{\sigma} \circ R(X,Y)) \end{aligned} \qquad (5)$$

for all $X, Y \in \Im_{0}^{1}(M)$ and $\omega, \theta \in \Im_{1}^{0}(M)$, where R is the Riemannian curvature of g defined by $R(X,Y) = [\nabla_{X}, \nabla_{Y}] - \nabla_{[X,Y]}$ (for more details, see [10]). If f is a differentiable function on M, $^{V}f = f \circ \pi$ denotes its canonical vertical lift to the coframe bundle $F^{*}(M)$.

Let (U, x^{i}) be a local coordinate system in M. In $U \subset M$, we put

$$X_{(i)} = \partial/(\partial x^{i}), \quad \theta^{(i)} = dx^{i}, i = 1,2,\ldots,n.$$

Taking into account (2) and (4), we see that

$$^{H}X^{(i)}=D_i=\begin{pmatrix}\delta_i^h\\ X_m^{\beta}\Gamma_{ih}^m\end{pmatrix}, \tag{6}$$

$$^{V_\alpha}\theta^{(i)}=D_{i_\alpha}=\begin{pmatrix}0\\ \delta_\beta^\alpha\delta_{ih}^i\end{pmatrix} \tag{7}$$

with respect to the natural frame $\{\partial_h,\partial_{h_\beta}\}$. It follows that this $n+n^2$ vector fields are linearly independent and generate, respectively the horizontal distribution of linear connection ∇ and the vertical distribution of linear coframe bundle $F^*(M)$. The set $\{D_H\}=\{D_h,D_{h_\beta}\}$ is called the frame adapted to linear connection ∇ on $\pi^{-1}(U)\subset F^*(M)$. From (2), (4), (6) and (7), we deduce that the horizontal lift ^{H}V of $V\in\Im_0^1(M)$ and vertical lift $^{V_\alpha}\omega$ of $\omega\in\Im_1^0(M)$ for each $\alpha=1,2,...,n,$ have respectively, components:

$$^{H}V=V^hD_h=\begin{pmatrix}V^h\\ 0\end{pmatrix}, \tag{8}$$

$$^{V_\alpha}\omega=\sum_h\omega_h\delta_\beta^\alpha D_{h_\beta}=\begin{pmatrix}0\\ \delta_\beta^\alpha\omega_h\end{pmatrix} \tag{9}$$

with respect to the adapted frame $\{D_I\}$. The non-holonomic objects $\Omega_{IJ}{}^K$ of the adapted frame $\{D_I\}$ are defined by

$$[D_I,D_J]=\Omega_{IJ}{}^KD_K$$

and have the following non-zero components:

$$\begin{cases}\Omega_{ij_\beta}{}^{k_\gamma}=-\Omega_{j_\beta i}{}^{k_\gamma}=-\delta_\beta^\gamma\Gamma_{ik}^j,\\ \Omega_{ij}{}^{k_\gamma}=X_m^\gamma R_{ijk}{}^m,\end{cases}$$

where $R_{ijk}{}^m$ local components of the Riemannian curvature R.

3. New class of metrics on the coframe bundle $F^*(M)$

Definition 3.1. *Let (M,g) be an $n-$dimensional Riemannian manifold and*

$f: M \to (0,+\infty)$ *be a positive smooth function on* M. *On the coframe bundle* $F^*(M)$, *we define a new class of Riemannian metrics* ${}^f g$ *by*

$$ {}^f g({}^H X, {}^H Y) = {}^V(g(X,Y)) = g(X,Y)\circ\pi, \tag{10} $$

$$ {}^S g_f({}^H X, {}^{V_\beta}\theta) = 0, \tag{11} $$

$$ {}^f g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = f\cdot\delta_{\alpha\beta} g^{-1}(\omega, X^\alpha) g^{-1}(\theta, X^\beta), \tag{12} $$

where $X, Y \in \Im_0^1(M)$ *and* $\omega, \theta \in \Im_1^0(M)$..

From (10)-(12) we determine that the metric ${}^f g$ has components

$$ {}^f g(D_i, D_j) = {}^V(g(\partial_i, \partial_j)) = g_{ij}, $$

$$ {}^f g(D_{i_\alpha}, D_j) = 0, $$

$$ {}^f g(D_{i_\alpha}, D_{j_\beta}) = f\cdot\delta_{\alpha\beta} g^{-1}(dx^i, X_r^\alpha) g^{-1}(dx^j, X_s^\beta) = f\cdot\delta_{\alpha\beta} g^{ir} g^{js} X_r^\alpha X_s^\beta $$

with respect to the adapted frame $\{D_I\}$ of coframe bundle $F^*(M)$.

From (3) and (4), it follows that the complete lift ${}^C X$ of $X \in \Im_0^1(M)$ is expressed by

$$ {}^C X - {}^H X = -X_m^\alpha \sum_i (\partial_i X^m - \Gamma_{ik}^m X^k)\partial_{i_\alpha} $$

$$ = -X_m^\alpha \sum_i \nabla_i X^m \partial_{i_\alpha} = -\delta_\alpha^\beta X_m^\alpha \nabla_i X^m \partial_{i_\beta} = -\sum_{\alpha=1}^n {}^{V_\alpha}(X_m^\alpha \nabla_i X^m), $$

i.e.,

$$ {}^C X = {}^H X - \sum_{\alpha=1}^n {}^{V_\alpha}(X^\alpha \circ \nabla X), \tag{13} $$

where

$$ X^\alpha \circ \nabla X = X_m^\alpha \nabla_i X^m dx^i. $$

Using (10)-(12) and (13), we get

$$ \begin{aligned} {}^f g({}^C X, {}^C Y) &= {}^f g\Big({}^H X - \sum_{\alpha=1}^n {}^{V_\alpha}(X^\alpha \circ \nabla X), {}^H Y - \sum_{\beta=1}^n {}^{V_\beta}(X^\beta \circ \nabla Y)\Big) \\ &= {}^V(g(X,Y)) + f\cdot \sum_{\alpha=1}^n \sum_{\beta=1}^n \delta_{\alpha\beta} g^{-1}(X^\alpha \circ \nabla X, X^\alpha) g^{-1}(X^\beta \circ \nabla Y, X^\beta),, \end{aligned} \tag{14} $$

where

$$g^{-1}(X^{\alpha} \circ \nabla X, X^{\alpha}) = g^{ir}(X_m^{\alpha} \circ \nabla X)_i X_r^{\alpha}.$$

Since the tensor field ${}^f g \in \Im_2^0(F^*(M))$ is completely determined also by its action on vector fields ${}^C X$ and ${}^C Y$, we have an alternative characterization of ${}^f g$ on $F^*(M)$: ${}^f g$ is completely determined by the condition (14).

It is known that the Riemannian metric defined by conditions (10) and (11) is called the natural metric (see, [3], [12]). Following definition , we conclude that metric ${}^f g$ is included in the class of natural metrics.

4. The Levi-Civita connection of ${}^f g$

Before studying the properties of Levi-Civita connection ${}^f \nabla$ of the coframe bundle $F^*(M)$ with the metric ${}^f g$, we prove the following theorems that are will be use later.

Theorem 4.1. *Let (M, g) be a Riemannian manifold and $\rho : R \to R$ be a smooth function. Then for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, we have*

1. ${}^H X(\rho(r_\alpha^2)) = 0;$
2. ${}^{V_\beta}\omega(\rho(r_\alpha^2)) = 2\delta_\alpha^\beta \rho'(r_\alpha^2) g^{-1}(\omega, X^\alpha);$
3. ${}^H X(g^{-1}(\theta, X^\alpha)) = g^{-1}(\nabla_X \theta, X^\alpha);$
4. ${}^{V_\beta}\omega(g^{-1}(\theta, X^\alpha)) = \delta_\alpha^\beta g^{-1}(\omega, \theta),$

where $r_\alpha^2 = g^{-1}(X^\alpha, X^\alpha).$

Proof. 1. Direct calculations using (4) give

$${}^H X(\rho(r_\alpha^2)) = (X^i D_i)(\rho(r_\alpha^2)) = X^i(\partial_i + X_r^\sigma \Gamma_{ij}^r \partial_{j_\sigma})(\rho(r_\alpha^2))$$

$$= X^i \rho'(r_\alpha^2)\partial_i(r_\alpha^2) + X^i \rho'(r_\alpha^2) X_r^\sigma \Gamma_{ij}^r \partial_{j_\sigma}(r_\alpha^2) = \rho'(r_\alpha^2)\Big[X^i \partial_i(g^{ks} X_k^\alpha X_s^\alpha)$$

$$+ X^i X_r^\sigma \Gamma_{ij}^r \partial_{j_\sigma}(g^{ks} X_k^\alpha X_s^\alpha)\Big] = \rho'(r_\alpha^2)\Big[X^i X_k^\alpha X_s^\alpha(-\Gamma_{il}^k g^{ls} - \Gamma_{il}^s g^{kl})$$

$$+ X^i X_r^\sigma \Gamma_{ij}^r g^{ks} X_s^\alpha \delta_\sigma^\alpha \delta_k^j + X^i X_r^\sigma \Gamma_{ij}^r g^{ks} X_k^{\ \alpha} \delta_\sigma^\alpha \delta_s^j \Big|$$

$$= \rho'(r_\alpha^2)\Big[-X^i X_k^\alpha X_s^\alpha \Gamma_{il}^k g^{ls} - X^i X_k^\alpha X_s^\alpha \Gamma_{il}^s g^{kl} + X^i X_r^\alpha X_s^\alpha \Gamma_{ik}^r g^{ks}$$

$+X^i X_r^\alpha X_k^\alpha \Gamma_{is}^r g^{ks} = 0;$

2. $^{V_\beta}\omega(\rho(r_\alpha^2)) = \sum_i \omega_i \delta_\sigma^\beta D_{i_\sigma}(\rho(r_\alpha^2)) = \rho'(r_\alpha^2)\sum_i \omega_i \delta_\sigma^\beta \partial_{i_\sigma}(g^{ks} X_k^\alpha X_s^\alpha)$

$= \rho'(r_\alpha^2)\sum_i \omega_i \delta_\sigma^\beta g^{ks}(X_s^\alpha \delta_\sigma^\alpha \delta_k^i + X_k^\alpha \delta_\sigma^\alpha \delta_s^i) = \rho'(r_\alpha^2)\sum_i \omega_i \delta_\alpha^\beta \cdot 2g^{is} X_s^\alpha$

$= 2\delta_\alpha^\beta \rho'(r_\alpha^2) g^{-1}(\omega, X^\alpha);$

3. $^H X(g^{-1}(\theta, X^\alpha)) = (X^i D_i)(g^{ks}\theta_k X_s^\alpha) = X^i \partial_i g^{ks}\theta_k X_s^\alpha$

$+X^i g^{ks}\partial_i \theta_k X_s^\alpha + X^i X_r^\sigma \Gamma_{ij}^r g^{ks}\theta_k \delta_\sigma^\alpha \delta_s^j = -X^i \Gamma_{il}^k g^{ls}\theta_k X_s^\alpha$

$-X^i \Gamma_{il}^s g^{kl}\theta_k X_s^\alpha + X^i g^{ks}\partial_i \theta_k X_s^\alpha + X^i X_r^\alpha \Gamma_{is}^r g^{ks}\theta_k = X^i g^{ks} X_s^\alpha(\partial_i \theta_k$

$-\Gamma_{ik}^l \theta_l) = X^i g^{ks} X_s^\alpha \nabla_i \theta_k = g^{-1}(\nabla_X \theta, X^\alpha);$

4. $^{V_\beta}\omega(g^{-1}(\theta, X^\alpha)) = \sum_i \omega_i \delta_\sigma^\beta D_{i_\sigma}(g^{ks}\theta_k X_s^\alpha) = \sum_i \omega_i \delta_\sigma^\beta \partial_{i_\sigma}(g^{ks}\theta_k X_s^\alpha)$

$= \sum_i \omega_i \delta_\sigma^\beta g^{ks}\theta_k \delta_\alpha^\sigma \delta_s^i = \sum_i \omega_i \delta_\alpha^\beta g^{ki}\theta_k = \delta_\alpha^\beta g^{-1}(\omega, \theta).$

Theorem 4.2. *Let* (M, g) *be a Riemannian manifold and* $(F^*(M), {}^f g)$ *its coframe bundle equipped with the metric* ${}^f g$*. Then for all* $X \in \Im_{\le 0}^1(M)$ *and* $\omega, \theta, \eta \in \Im_1^0(M)$*, we have*

1. $^H X({}^f g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = \frac{1}{f} X(f)\, {}^f g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) + {}^f g({}^{V_\alpha}(\nabla_X \omega), {}^{V_\beta}\theta)$

$+ {}^f g({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_X \theta));$

2. $^{V_\alpha}\omega({}^f g({}^{V_\beta}\theta, {}^{V_\gamma}\eta)) = f\delta_{\alpha\gamma} g^{-1}(\omega, \theta) g^{-1}(\eta, X^\gamma)$

$+ f\delta_{\beta\alpha} g^{-1}(\theta, X^\beta) g^{-1}(\omega, \eta).$

Proof. The proof of Theorem 4.2 directly follows from Theorem 4.1:

1. $^H X({}^f g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = {}^H X(f\delta_{\alpha\beta} g^{-1}(\omega, X^\alpha) g^{-1}(\theta, X^\beta))$

$= X(f)\delta_{\alpha\beta} g^{-1}(\omega, X^\alpha) g^{-1}(\theta, X^\beta) + f\delta_{\alpha\beta} g^{-1}(\nabla_X \omega, X^\alpha) g^{-1}(\theta, X^\beta)$

$+ f\delta_{\alpha\beta} g^{-1}(\omega, X^\alpha) g^{-1}(\nabla_X \theta, X^\beta) = \frac{1}{f} X(f)\, {}^f g({}^{V_\alpha}\omega, {}^{V_\beta}\theta)$

$+{}^{f}g({}^{V_\alpha}(\nabla_X\omega),{}^{V_\beta}\theta)+{}^{f}g({}^{V_\alpha}\omega,{}^{V_\beta}(\nabla_X\theta));$

2. ${}^{V_\alpha}\omega({}^{f}g({}^{V_\beta}\theta,{}^{V_\gamma}\eta))={}^{V_\alpha}\omega(f\delta_{\beta\gamma}g^{-1}(\omega,X^\beta)g^{-1}(\eta,X^\gamma)$

$=\omega(f)\delta_{\beta\gamma}g^{-1}(\theta,X^\beta)g^{-1}(\eta,X^\gamma)+f\delta_{\beta\gamma}{}^{V_\alpha}\omega(g^{-1}(\theta,X^\beta))g^{-1}(\eta,X^\gamma)$

$+f\delta_{\beta\gamma}g^{-1}(\theta,X^\beta){}^{V_\alpha}\omega(g^{-1}(\eta,X^\gamma))=f\delta_{\beta\gamma}\delta^\alpha_\beta g^{-1}(\omega,\theta)g^{-1}(\eta,X^\gamma)$

$+f\delta_{\beta\gamma}\delta^\alpha_\gamma g^{-1}(\theta,X^\beta)g^{-1}(\omega,\eta)=f\delta_{\alpha\gamma}g^{-1}(\omega,\theta)g^{-1}(\eta,X^\gamma)$

$+f\delta_{\beta\alpha}g^{-1}(\theta,X^\beta)g^{-1}(\omega,\eta).$

Based on Theorem 4.1 and Theorem 4.2, we prove the following theorem on the Levi-Civita connection ${}^{f}\nabla$ of the coframe bundle $F^*(M)$ with the metric ${}^{f}g$.

Theorem 4.3. *Connection ${}^{f}\nabla$ satisfies the following relations:*

$i)\ {}^{f}\nabla_{{}^HX}{}^HY={}^H(\nabla_XY),$

$ii)\ {}^{f}\nabla_{{}^HX}{}^{V_\beta}\theta={}^{V_\beta}(\nabla_X\theta)+\frac{1}{2f}X(f){}^{V_\beta}\theta,$

$$iii)\ {}^{f}\nabla_{{}^{V_\alpha}\omega}{}^HY=\frac{1}{2f}Y(f){}^{V_\alpha}\omega, \tag{15}$$

$iv)\ {}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta=0$ for $\alpha\neq\beta$,

$${}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta=-g^{-1}(\omega,X^\alpha)g^{-1}(\theta,X^\alpha){}^H(gradf)+\tfrac{2}{r_\alpha^2}g^{-1}(\omega,\theta){}^{V_\alpha}X^\alpha,$$

for all $X,Y\in\Im^1_0(M),\omega,\theta\in\Im^0_1(M).$

Proof. The Levi-Civita connection ${}^{f}\nabla$ of coframe bundle $F^*(M)$ with Riemannian metric ${}^{f}g$ is characterized by the Koszul formula

$$2{}^Sg_f(\tilde\nabla_{\tilde X}\tilde Y,\tilde Z)=\tilde X({}^Sg_f(\tilde Y,\tilde Z))+\tilde Y({}^Sg_f(\tilde Z,\tilde X))-\tilde Z({}^Sg_f(\tilde X,\tilde Y))$$
$$-{}^Sg_f(\tilde X,[\tilde Y,\tilde Z])+{}^Sg_f(\tilde Y,[\tilde Z,\tilde X])+{}^Sg_f(\tilde Z,[\tilde X,\tilde Y]) \tag{16}$$

for all vector fields $\tilde X,\tilde Y,\tilde Z\in\Im^1_0(M).$

$i)$ Direct calculations using (4), (5) and (16) give

$2{}^fg({}^f\nabla_{{}^HX}{}^HY,{}^HZ)={}^HX({}^fg({}^HY,{}^HZ))+{}^HY({}^fg({}^HZ,{}^HX))$

$$-{}^{H}Z({}^{f}g({}^{H}X,{}^{H}Y))-{}^{f}g({}^{H}X,[{}^{H}Y,{}^{H}Z])+{}^{f}g({}^{H}Y,[{}^{H}Z,{}^{H}X])$$
$$+{}^{f}g({}^{H}Z,[{}^{H}X,{}^{H}Y])=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))$$
$$-{}^{f}g({}^{H}X,{}^{H}[Y,Zy,{}^{H}[Y,Z]+\gamma R(Y,Z))+{}^{f}g({}^{H}Y,{}^{H}[Z,X]+\gamma R(Z,X))$$
$$+{}^{f}g({}^{H}Z,{}^{H}[X,Y]+\gamma R(X,Y))=X(g(Y,Z))+Y(g(Z,X))$$
$$-Z(g(X,Y))-g(X,[Y,Z])+g(Y,[Z,X])+g(Z,[X,Y])=2g(\nabla_{X}Y,Z)$$
$$=2\,{}^{f}g({}^{H}(\nabla_{\partial X}Y),{}^{H}Z),$$

and

$$2\,{}^{f}g({}^{f}\nabla_{{}^{H}X}{}^{H}Y,{}^{V_\gamma}\xi)={}^{H}X({}^{f}g({}^{H}Y,{}^{V_\gamma}\xi))+{}^{H}Y({}^{f}g({}^{V_\gamma}\xi,{}^{H}X))$$
$$-{}^{V_\gamma}\xi({}^{f}g({}^{H}X,{}^{H}Y))-{}^{f}g({}^{H}X,[{}^{H}Y,{}^{V_\gamma}\xi])+{}^{f}g({}^{H}Y,[{}^{V_\gamma}\xi,{}^{H}X])$$
$$+{}^{f}g({}^{V_\gamma}\xi,[{}^{H}X,{}^{H}Y])=-{}^{f}g({}^{H}X,{}^{V_\gamma}(\nabla_{Y}\xi))+{}^{f}g({}^{H}Y,-{}^{V_\gamma}(\nabla_{X}\xi))$$
$$+{}^{f}g({}^{V_\gamma}\xi,{}^{H}[X,Y]+\gamma R(X,Y))={}^{f}g({}^{V_\gamma}\xi,\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^{\sigma}\circ R(X,Y)))$$
$$=\sum_{\sigma=1}^{n}{}^{f}g({}^{V_\gamma}\xi,{}^{V_\sigma}(X^{\sigma}\circ R(X,Y)))$$
$$=\sum_{\sigma=1}^{n}\delta_{\gamma\sigma}fg^{-1}(\xi,X^{\gamma})g^{-1}(X^{\sigma}\circ R(X,Y),X^{\sigma})=0,$$

from which it follows that

$${}^{f}\nabla_{{}^{H}X}{}^{H}Y={}^{H}(\nabla_{X}Y).$$

$ii)$ Calculations similar to those in $i)$ give

$$2\,{}^{f}g({}^{f}\nabla_{{}^{H}X}{}^{V_\beta}\theta,{}^{H}Z)={}^{H}X({}^{f}g({}^{V_\beta}\theta,{}^{H}Z))+{}^{V_\beta}\theta({}^{f}g({}^{H}Z,{}^{H}X))$$
$$-{}^{H}Z({}^{f}g({}^{H}X,{}^{V_\beta}\theta))-{}^{f}g({}^{H}X,[{}^{V_\beta}\theta,{}^{H}Z])+{}^{f}g({}^{V_\beta}\theta,[{}^{H}Z,{}^{H}X])$$
$$+{}^{f}g({}^{H}Z,[{}^{H}X,{}^{V_\beta}\theta])={}^{V_\beta}\theta(g(Z,X))+{}^{f}g({}^{V_\beta}\theta,{}^{H}[Z,X]$$
$$+\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^{\sigma}\circ R(Z,X)))={}^{f}g({}^{V_\beta}\theta,\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^{\sigma}\circ R(Z,X)))$$
$$=\sum_{\sigma=1}^{n}\delta_{\beta\sigma}fg^{-1}(\theta,X^{\beta})g^{-1}(X^{\sigma}\circ R(Z,X),X^{\sigma})=0.$$

Also by help of (5),(8), (9) and (16), we obtain:

$$2\,{}^f g({}^f\nabla_{{}^H X}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = {}^H X({}^f g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^{V_\beta}\theta({}^f g({}^{V_\gamma}\xi, {}^H X))$$
$$-{}^{V_\gamma}\xi({}^f g({}^H X, {}^{V_\beta}\theta)) - {}^f g({}^H X, [{}^{V_\beta}\theta, {}^{V_\gamma}\xi]) + {}^f g({}^{V_\beta}\theta, [{}^{V_\gamma}\xi, {}^H X])$$
$$+{}^f g({}^{V_\gamma}\xi, [{}^H X, {}^{V_\beta}\theta]) = {}^H X({}^f g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^f g({}^{V_\beta}(\nabla_X\theta), {}^{V_\gamma}\xi))$$
$$+{}^f g({}^{V_\beta}\theta, {}^{V_\gamma}(\nabla_X\xi)).$$

Using the first formula of Theorem 4.2, we have

$$2\,{}^f g({}^f\nabla_{{}^H X}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = \tfrac{1}{f}X(f)\,{}^f g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^f g({}^{V_\beta}(\nabla_X\theta), {}^{V_\gamma}\xi)$$
$$+{}^f g({}^{V_\beta}\theta, {}^{V_\gamma}(\nabla_X\xi)) + {}^f g({}^{V_\beta}(\nabla_X\theta), {}^{V_\gamma}\xi) - {}^f g({}^{V_\beta}\theta, {}^{V_\gamma}(\nabla_X\xi))$$
$$= \tfrac{1}{f}X(f)\,{}^f g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + 2\,{}^f g({}^{V_\beta}(\nabla_X\theta), {}^{V_\gamma}\xi),$$

from which it follows that

$${}^f\nabla_{{}^H X}{}^{V_\beta}\theta = {}^{V_\beta}(\nabla_X\theta) + \frac{1}{2f}X(f)\,{}^{V_\beta}\theta.$$

iii) Direct calculations using (5),(8),(9) and (16) give

$$2\,{}^S g_f({}^f\nabla_{{}^{V_\alpha}\omega}{}^H Y, {}^H Z) = {}^{V_\alpha}\omega({}^f g({}^H Y, {}^H Z)) + {}^H Y({}^f g({}^H Z, {}^{V_\alpha}\omega))$$
$$-{}^H Z({}^f g({}^{V_\alpha}\omega, {}^H Y)) - {}^f g({}^{V_\alpha}\omega, [{}^H Y, {}^H Z]) + {}^f g({}^H Y, [{}^H Z, {}^{V_\alpha}\omega])$$
$$+{}^f g({}^H Z, [{}^{V_\alpha}\omega, {}^H Y]) = {}^{V_\alpha}\omega(g(Y,Z)) - {}^f g({}^{V_\alpha}\omega, {}^H[Y,Z]$$
$$+\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(Y,Z))) = -{}^f g({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(Y,Z)))$$
$$-\sum_{\omega=1}^{n}{}^f g({}^{V_\alpha}\omega, {}^{V_\sigma}(X^\sigma \circ R(Y,Z)))$$
$$= -f\sum_{\omega=1}^{n}\delta_{\alpha\sigma}g^{-1}(\omega, X^\alpha)g^{-1}(X^\sigma \circ R(Y,Z), X^\sigma)) = 0,$$

and

$$2\,{}^S g_f({}^f\nabla_{{}^{V_\alpha}\omega}{}^H Y, {}^{V_\gamma}\xi) = {}^{V_\alpha}\omega({}^f g({}^H Y, {}^{V_\gamma}\xi)) + {}^H Y({}^f g({}^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$
$$-{}^{V_\gamma}\xi({}^f g({}^{V_\alpha}\omega, {}^H Y)) - {}^f g({}^{V_\alpha}\omega, [{}^H Y, {}^{V_\gamma}\xi]) + {}^f g({}^H Y, [{}^{V_\gamma}\xi, {}^{V_\alpha}\omega])$$
$$+{}^f g({}^{V_\gamma}\xi, [{}^{V_\alpha}\omega, {}^H Y]) = {}^f g({}^{V_\gamma}(\nabla_Y\xi), {}^{V_\alpha}\omega) + {}^f g({}^{V_\gamma}\xi, {}^{V_\alpha}(\nabla_Y\omega))$$

$$-{}^{f}g({}^{V_\gamma}\xi, {}^{V_\alpha}(\nabla_Y\omega)) - {}^{f}g({}^{V_\alpha}\omega, {}^{V_\gamma}(\nabla_Y\xi)) + \tfrac{1}{f}Y(f)\,{}^{f}g({}^{V_\alpha}\omega, {}^{V_\gamma}\xi)$$

$$+, \tfrac{1}{f}Y(f)\,{}^{f}g({}^{V_\alpha}\omega, {}^{V_\gamma}\xi),$$

From which we deduce that

$${}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{H}Y = \tfrac{1}{2f}Y(f)\,{}^{V_\alpha}\omega.$$

$iv)$ Direct calculations using (5),(8),(9) and (16) give

$$2\,{}^{S}g_f({}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta, {}^{H}Z) = {}^{V_\alpha}\omega({}^{f}g({}^{V_\beta}\theta, {}^{H}Z)) + {}^{V_\beta}\theta({}^{f}g({}^{H}Z, {}^{V_\alpha}\omega))$$

$$-{}^{H}Z({}^{f}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) - {}^{f}g({}^{V_\alpha}\omega, [{}^{V_\beta}\theta, {}^{H}Z]) + {}^{f}g({}^{V_\beta}\theta, [{}^{H}Z, {}^{V_\alpha}\omega])$$

$$+{}^{f}g({}^{H}Z, [{}^{V_\alpha}\omega, {}^{V_\beta}\theta]) = -{}^{f}g({}^{V_\alpha}(\nabla_Z\omega), {}^{V_\beta}\theta) - {}^{f}g({}^{V_\alpha}\omega{}^{V_\beta}(\nabla_Z\theta))$$

$$+{}^{f}g({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Z\theta)) - {}^{f}g({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Z\omega)) - \tfrac{1}{f}Z(f)\,{}^{f}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta)$$

$$= -Z(f)\delta_{\alpha\beta}g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\beta)$$

$$= -\delta_{\alpha\beta}g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\beta)\,{}^{f}g({}^{H}(gradf), {}^{H}Z),$$

where

$${}^{f}g({}^{H}(gradf), {}^{H}Z) = g(gradf, Z) = Z(f).$$

On the other hand, direct calculations give

$$2\,{}^{S}g_f({}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = {}^{V_\alpha}\omega({}^{f}g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^{V_\beta}\theta({}^{f}g({}^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$

$$-{}^{V_\gamma}\xi({}^{f}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) - {}^{f}g({}^{V_\alpha}\omega, [{}^{V_\beta}\theta, {}^{V_\gamma}\xi]) + {}^{f}g({}^{V_\beta}\theta, [{}^{V_\gamma}\xi, {}^{V_\alpha}\omega])$$

$$+{}^{f}g({}^{V_\gamma}\xi, [{}^{V_\alpha}\omega, {}^{V_\beta}\theta]) = {}^{V_\alpha}\omega({}^{f}g({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^{V_\beta}\theta({}^{f}g({}^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$

$$-{}^{V_\gamma}\xi({}^{f}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) = f\delta_{\alpha\gamma}g^{-1}(\omega,\theta)g^{-1}(\xi, X^\gamma)$$

$$+f\delta_{\beta\alpha}g^{-1}(\theta, X^\beta)g^{-1}(\omega,\xi) + f\delta_{\beta\alpha}g^{-1}(\theta,\xi)g^{-1}(\omega, X^\alpha)$$

$$+f\delta_{\gamma\beta}g^{-1}(\xi, X^\gamma)g^{-1}(\theta,\omega) - f\delta_{\gamma\beta}g^{-1}(\xi,\omega)g^{-1}(\theta, X^\beta)$$

$$-f\delta_{\alpha\gamma}g^{-1}(\omega, X^\alpha)g^{-1}(\xi,\theta).$$

If we put $\gamma \neq \beta \neq \alpha,$ then

$$2\,{}^{S}g_f({}^{f}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = 0,$$

from which it follows that

$$^f\nabla_{^{V_\alpha}\omega}{}^{V_\beta}\theta = 0 \text{ for } \alpha \neq \beta.$$

If we put $\gamma = \beta = \alpha$. Calculations like above give

$$2\,{}^S g_f({}^f\nabla_{^{V_\alpha}\omega}{}^{V_\alpha}\theta, {}^{V_\alpha}\xi) = fg^{-1}(\omega,\theta)g(\xi, X^\alpha)$$
$$+fg^{-1}(\theta, X^\alpha)g^{-1}(\omega,\xi) + fg^{-1}(\theta,\xi)g^{-1}(\omega, X \ \ )$$
$$+fg^{-1}(\xi, X^\alpha)g^{-1}(\theta,\omega) - + fg^{-1}(\xi,\omega)g^{-1}(\theta, X^\alpha)$$
$$-fg^{-1}(\omega, X^\alpha)g^{-1}(\xi,\theta) = 2fg^{-1}(\omega,\theta)g^{-1}(\xi, X^\alpha)$$
$$= \tfrac{2}{r_\alpha^2} g^{-1}(\omega,\theta)\,{}^f g({}^{V_\alpha}X^\alpha, {}^{V_\alpha}\xi),$$

from which it follows that

$$^f\nabla_{^{V_\alpha}\omega}{}^{V_\alpha}\theta = -g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha)^H(gradf) + \tfrac{1}{r_\alpha^2} g^{-1}(\omega,\theta)^{V_\alpha}X^\alpha.$$

Hence theorem is proved.

5. Components of $^f\nabla$

We write

$$^f\nabla_{D_I}D_J = {}^f\Gamma_{IJ}^K D_K \tag{16}$$

with respect to the adapted frame $\{D_K\}$ of linear coframe bundle $F^*(M)$, where $^f\Gamma_{IJ}^K$ denote the components (Christoffel symbols) of Levi-Civita connection $^{f\cdot}\nabla$. Then by using of Theorem 4.3, we get following.

Theorem 5.1. *Let (M, g) be a Riemannian manifold and $^f\nabla$ be the Levi-Civita connection of the linear coframe bundle $F^*(M)$ with the metric $^f g$. Then particular values of $^f\Gamma_{IJ}^K$ for different indices by taking account of (16) are then found to be*

$$^f\Gamma_{ij}^k = \Gamma_{ij}^k, \qquad {}^f\Gamma_{ij}^{k_\gamma} = \Gamma_{ij_\beta}^k = 0,$$
$$^f\Gamma_{ij_\beta}^{k_\gamma} = \tfrac{1}{2f}\delta_\gamma^\beta f_i\delta_k^j -, \delta_\gamma^\beta\Gamma_{ik}^j, \qquad {}^f\Gamma_{i_\alpha j}^k = 0,$$
$$^f\Gamma_{i_\alpha j}^{k_\gamma} = \tfrac{1}{2f}\delta_\gamma^\alpha f_j\delta_k^i, \qquad {}^f\Gamma_{i_\alpha j_\beta}^k = {}^f\Gamma_{i_\alpha j_\beta}^{k_\gamma} = 0 \quad \textit{for } \alpha \neq \beta,$$
$$^f\Gamma_{i_\alpha j_\alpha}^k = -g^{is}g^{lj}g^{kr}X_s^\alpha X_l^\alpha f_r, \qquad {}^f\Gamma_{i_\alpha j_\alpha}^{k_\gamma} = \tfrac{2}{r_\alpha 2} g^{ij}\delta_\gamma^\alpha X_k^\alpha.$$

Proof. Let $X,Y\in\Im_0^1(M),\omega,\theta\in\Im_1^0(M)$. Using formulas (15) and (16), we obtain:

$$^{f}\nabla_{^{H}X}{}^{H}Y={}^{f}\nabla_{X^{i}D_{i}}(Y^{j}D_{j})=X^{i}\,{}^{f}\nabla_{D_{i}}(Y^{j}D_{j}) \tag{17}$$

$$=X^{i}\,{}^{f}(Y^{j}\,{}^{f}\nabla_{D_{i}}D_{j}+D_{i}Y^{j}D_{j})=X^{i}Y^{j}\,{}^{f}\Gamma_{ij}^{k}D_{k}$$

$$+X^{i}Y^{j}\,{}^{f}\Gamma_{ij}^{k_{\gamma}}D_{k_{\gamma}}+X^{i}\partial_{i}Y^{j}D_{j},$$

and

$$^{H}(\nabla_{X}Y)=(\nabla_{X}Y)^{i}D_{i}=X^{j}\partial_{j}Y^{i}D_{i}+X^{j}Y^{s}\Gamma_{js}^{i}D_{i}. \tag{18}$$

Equating the right-hand sides of equalities (17) and (18), we will have

$$^{f}\Gamma_{ij}^{k}=\Gamma_{ij}^{k},\qquad {}^{f}\Gamma_{ij}^{k_{\gamma}}=0.$$

Similarly, direct calculations using (15) and (16) give

$$^{f}\nabla_{^{H}X}{}^{V_{\beta}}\theta={}^{f}\nabla_{X^{i}D_{i}}(\delta_{\sigma}^{\beta}\theta_{j}D_{j_{\sigma}})=X^{i}\,{}^{f}\nabla_{D_{i}}(\delta_{\sigma}^{\beta}\theta_{j}D_{j_{\sigma}})$$

$$=\delta_{\sigma}^{\beta}X^{i}(D_{i}\theta_{j}D_{j_{\sigma}}+\theta_{j}\nabla_{D_{i}}D_{j_{\sigma}})=\delta_{\sigma}^{\beta}X^{i}\partial_{i}\theta_{j}D_{j_{\sigma}}$$

$$+\delta_{\sigma}^{\beta}X^{i}\theta_{j}\,{}^{f}\Gamma_{ij_{\sigma}}^{k}D_{k}+\delta_{\sigma}^{\beta}X^{i}\theta_{j}\,{}^{f}\Gamma_{ij_{\sigma}}^{k_{\gamma}}D_{k}, \tag{19}$$

and

$$^{V_{\beta}}(\nabla_{X}\theta)+\tfrac{1}{2f}X(f)^{V_{\beta}}\theta=\delta_{\sigma}^{\beta}(X^{i}(\partial_{i}\theta_{j}-\Gamma_{ij}^{m}\theta_{m}))D_{j_{\sigma}}$$

$$+\tfrac{1}{2f}X^{i}\partial_{i}f\,\delta_{\sigma}^{\beta}\theta_{j}D_{j_{\sigma}}=\delta_{\sigma}^{\beta}X^{i}\partial_{i}\theta_{j}D_{j_{\sigma}}-\delta_{\sigma}^{\beta}X^{i}{}_{j}\Gamma_{ij}^{m}\theta_{m}D_{j_{\sigma}}$$

$$+\tfrac{1}{2f}X^{i}f_{i}\delta_{\sigma}^{\beta}\theta_{j}D_{j_{\sigma}}. \tag{20}$$

Comparing the right-hand sides of equalities (19) and (20), we arrive at the following

$$^{f}\Gamma_{ij_{\beta}}^{k}=0,\qquad {}^{f}\Gamma_{ij_{\beta}}^{k_{\gamma}}=-\delta_{\gamma}^{\beta}f_{i}\delta_{k}^{j},$$

where $f_i=\frac{\partial f}{\partial x^i}$.

By calculations similar to those above we yield

$$^{f}\nabla_{^{V_{\alpha}}\omega}{}^{H}Y={}^{f}\nabla_{\delta_{\sigma}^{\alpha}\omega_{i}D_{i_{\varpi}}}(Y^{j}D_{j})=\delta_{\sigma}^{\alpha}\omega_{i}\,{}^{f}\nabla_{D_{i_{\varpi}}}(Y^{j}D_{j})$$

$$=\delta_{\sigma}^{\alpha}\omega_{i}Y^{j}\,{}^{f}\nabla_{D_{i_{\varpi}}}D_{j}=\delta_{\sigma}^{\alpha}\omega_{i}Y^{j}\,{}^{f}\Gamma_{i_{\sigma}j}^{K}D_{K}=\delta_{\sigma}^{\alpha}\omega_{i}Y^{j}\,{}^{f}\Gamma_{i_{\sigma}j}^{k}D_{k}$$

$$+\delta_\sigma^\alpha \omega_i Y^j \, {}^f\Gamma_{i_\sigma j}^{k_\gamma} D_{k_\gamma}, \tag{21}$$

and

$$\tfrac{1}{2f} Y(f)^{V_\alpha}\omega = \tfrac{1}{2f} Y^j \partial_j f \delta_\sigma^\alpha \omega_i D_{i_\sigma} = \tfrac{1}{2f} Y^j f_j \delta_\sigma^\alpha \omega_i D_{i_\sigma}. \tag{22}$$

Comparing the right-hand sides of equalities (21) and (22), we deduce followings:

$${}^f\Gamma_{i_\alpha j}^{k} = 0, \qquad {}^f\Gamma_{i_\alpha j}^{k_\gamma} = \tfrac{1}{2f}\delta_\gamma^\alpha f_j \delta_k^i.$$

Now we assume that $\alpha \neq \beta.$ Then by using (9),(15) and (16), we have

$${}^f\nabla_{{}^{V_\alpha}\omega} {}^{V_\beta}\theta = {}^f\nabla_{\delta_\sigma^\alpha \omega_i D_{i_\sigma}} (\delta_\tau^\beta \theta_j D_{j_\tau}) = \delta_\sigma^\alpha \omega_i \, {}^f\nabla_{D_{i_\sigma}} (\delta_\tau^\beta \theta_j D_{j_\tau})$$

$$= \delta_\sigma^\alpha \omega_i \delta_\tau^\beta \theta_j \, {}^f\nabla_{D_{i_\sigma}} D_{j_\tau} = \delta_\sigma^\alpha \omega_i \delta_\tau^\beta \theta_j \, {}^f\Gamma_{i_\sigma j_\tau}^{k} D_k$$

$$+\delta_\sigma^\alpha \omega_i \delta_\tau^\beta \theta_j \, {}^f\Gamma_{i_\sigma j_\tau}^{k_\gamma} D_{k_\gamma} = 0.$$

The last relation shows that

$${}^f\Gamma_{i_\alpha j_\beta}^{k} = 0, \qquad {}^f\Gamma_{i_\alpha j_\beta}^{k_\gamma} = 0 \quad for \ \alpha \neq \beta.$$

If $\alpha = \beta.$ Direct calculations like above give

$${}^f\nabla_{{}^{V_\alpha}\omega} {}^{V_\alpha}\theta = \delta_\sigma^\alpha \omega_i \delta_\tau^\alpha \theta_j \, {}^f\Gamma_{i_\sigma j_\tau}^{k} D_k + \delta_\sigma^\alpha \omega_i \delta_\tau^\alpha \theta_j \, {}^f\Gamma_{i_\sigma j_\tau}^{k_\gamma} D_{k_\gamma}$$

$$= \omega_i \theta_j \, {}^f\Gamma_{i_\alpha j_\alpha}^{k} D_k + \omega_i \theta_j \, {}^f\Gamma_{i_\alpha j_\alpha}^{k_\gamma} D_{k_\gamma} = -g^{rs}\omega_r X_s^\alpha g^{lm}\theta_m X_l^\alpha (g^{ir} f_r) D_i$$

$$+\tfrac{2}{r_\alpha^2} g^{rs}\omega_r \theta_s \delta_\sigma^\alpha X_l^\alpha D_{l_\sigma} = \omega_i \theta_j (-g^{is} g^{lj} X_s^\alpha X_l^\alpha g^{kr} f_r D_k$$

$$+\tfrac{2}{r_\alpha^2} g^{ij} \delta_\gamma^\alpha X_k^\alpha \omega_i \theta_j D_{k_\gamma},$$

from which it follows that:

$${}^f\Gamma_{i_\alpha j_\alpha}^{k} = -g^{is} g^{lj} X_s^\alpha X_l^\alpha g^{kr} f_r, \qquad {}^f\Gamma_{i_\alpha j_\alpha}^{k_\gamma} = \tfrac{2}{r_\alpha^2} g^{ij} \delta_\gamma^\alpha X_k^\alpha.$$

This completes the proof.

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